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SECOND VARIATION CONDITIONS FOR PROBLEMS WITH PARAMETERIZED CONTROL

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To My Grandparents, John and Maud Correa

SECOND VARIATION CONDITIONS FOR PROBLEMS WITH PARAMETERIZED CONTROL

by

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DISSERTATION

Presented to the Faculty of the Graduate School of

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CHRISTOPHER NOEL D'SOUZA

The University of Texas at Austin

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SECOND VARIATION CONDITIONS FOR PROBLEMS WITH PARAMETERIZED CONTROL

Publication No. ___________

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The University of Texas at Austin, 1991

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The second variation sufficient conditions for minima for optimal parameterized control problems are developed. These conditions are derived directly from the second variation by introducing a new variable μ whose differential equation has the same form as the differential equation for λ. The neighboring extremal trajectories are derived using the sweep method resulting in a set of Riccati equations. In addition, the neighboring extremal parameterized control law is derived. This is also done for the case when there are both parameterized and nonparameterized controls.

A perfect differential is added to the second variation which reduces to a form involving the neighboring extremal parameterized control law. The

second variation can thereby be reduced to a perfect square involving the neighboring extremal parameters. The second variation sufficient conditions are formulated using these results.

Second variation conditions are developed for the class of problems in which there are both parameterized and nonparameterized controls. This development closely follows the case of strictly parameterized control.

The second variation conditions are used to solve a missile intercept problem with a parameterized control. The model used is an EMRAAT (Enhanced Medium Range Air-to-Air Technology) class of missile with a single thrust phase. The control, which is the coefficient of lift, is parameterized and the control nodes are the parameters which which maximize the terminal velocity. Optimal trajectories for several realistic scenarios are obtained using a shooting method. The sufficient conditions are applied to these optimal trajectories and are shown to be minimal.

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List of Symbols

English Symbols

A,B,C,D,E,F	Riccati equation coefficients
$\tilde{A}$,$\tilde{C}$,$\tilde{D}$,$\tilde{E}$,$\tilde{F}$	Riccati equation coefficients
a,b,c	Partitioned elements of V^{-1}
f	Dynamical equations
G	Endpoint function
H	Hamiltonian
J	Performance Index
$\bar{J}$	Adjoined performance index
M,N,U,V	Matrices containing swwep variables
P,$\tilde{P}$	Perfect differentials
p	Parameters
S,Q,R,m,n	Backward sweep matrices
t	Dimensional time
u	Control variables
u_i	Parameterized control variables
x	State variables

Greek Symbols

α,β,γ	Backward sweep matrices

λ	Lagrange multipliers associated with the state variables
μ	'costate' variable
ν	Lagrange multipliers associated with the terminal constraints
τ	Nondimensional time
ϕ	Terminal penalty
ψ	Terminal constraints

English Symbols Used in Chapter 5

C_D	Coefficient of drag
C_{D_0}	Coefficient of drag at zero lift
C_D	Coefficient of Lift
D	Drag (lb)
g	Gravity (ft/sec^2)
h	Altitude (ft)
h_s	Scale height (ft)
h_{T_0}	Initial target altitude
K	Parabolic drag polar coefficient
L	Lift (lb)
m	Mass ($slugs$)
S	Reference area (ft^2)
T	Thrust (lb)
V	Velocity (ft/sec)
V_T	Target velocity (ft/sec)
x	Downrange position (ft)
x_{T_0}	Initial target downrange position

Greek Symbols Used in Chapter 5

γ	Flight Path Angle (rad)
γ_T	Target Flight Path Angle (rad)
λ_x,λ_h,λ_V,λ_γ	Lagrange multipliers associated with the state
μ_{C_L},μ_{t_f}	'Costate' variable associated with the parameters
ν_x,ν_h	Lagrange multipliers associated with the terminal constraints
ρ	Density ($slugs/ft^3$)
ρ_0	Density at sea level ($slugs/ft^3$)

Other Symbols

$\dot{( )}$	Derivative with respect to time
$( )'$	Derivative with respect to τ
$( )_p$	Partial of ( ) with respect to p
$( )_u$	Partial of ( ) with respect to u
$( )_x$	Partial of ( ) with respect to x
$( )_f$	Final value
$( )_o$	Initial value
$\delta( )$	Variation of ( )

Chapter 1

Introduction

During the past 30 years, many contributions have been made to the theory of deterministic optimal control theory. Much effort has been devoted to deriving necessary conditions and sufficient conditions for minima. The sufficient conditions are referred to as the conjugate point or Jacobi conditions in the classical calculus of variations literature [1].

The earliest work in deriving these conditions for optimal control problems was done by Breakwell et. al. [2]. The second variation was used to derive a neighboring optimal control law using both continuous and sampled data. Soon thereafter, the conjugate point conditions were derived from a control-theoretic viewpoint by Breakwell and Ho [3]. The accessory minimum problem was formulated and issues such as normality and controllability were addressed.

Bryson and Ho [4] summarized a great deal of the prior work in the field. It included the sufficient conditions for free-final time problems. However, these conditions could not be applied to a number of problems for the case where Q is noninvertible (as for the Zermelo problem).

Dyer and McReynolds [5] summarized much of the previous work done in the theory and computation of optimal control. They approached the problems from a dynamic programming framework. They included dynamic

programming theory dealing with control parameters. Many algorithms were included in this work.

Jacobson and Mayne [6] approached optimal control from a differential dynamic programming viewpoint. They described this class of algorithms and applied them to problems involving bang-bang control as well as discrete-time and stochastic systems

Wood and Bryson [7] presented another set of sufficient conditions for fixed final time problems. However, these conditions were rather cumbersome to apply to optimal trajectory problems. While these conditions were an improvement over those presented in [4], they were still difficult to apply because they involved identifying *a priori* the independent and dependent parts of the constraints and they involved integrating two sets of similar Ricatti equations corresponding to the different parts of the constraints. In addition, they could not be applied to the class of problems where the matrix Q is noninvertible.

Recently, Hull [8] obtained a set of sufficient conditions for problems with free-final time which are easy to apply and do not have the drawbacks of Refs. 4 and 7. A new matrix V is defined whose inverse existed for the case where Q is noninvertible. These conditions represent a substantial improvement over those presented earlier and have been applied and tested on a wide variety of problems, including a number of 'degenerate' non-singular deterministic optimal control problems (such as the Zermelo problem). In addition, Hull [9] presented a discussion of the variational process used in setting up and solving this class of problems.

A common technique of solving optimal control problems is to parameterize the control and assume that the control is piecewise linear and

continuous throughout the trajectory. This is the so-called 'suboptimal' control technique of solving optimal control problems. If a sufficiently dense set of nodes is chosen for the control, this class of problems yields solutions quite close to the optimal solution. The 'density' of the control nodes depends on the type of problem and the desired accuracy of the solution. As might be expected, the more numerous the nodes, the more computation time it takes for a solution. The advantage of this type of solution is that the optimal control problem is transformed into a parameter optimization problem involving a performance index to be minimized subject to a set of equality and inequality constraints. A number of nonlinear programming algorithms exist for the solution of this class of problems, such as VF02AD and GRG2. While it is generally believed that these algorithms give an optimal solution, it has been difficult to show that the solutions are indeed minima (or maxima).

Recently, Hull and Sheen [10] have introduced a 'parameterized shooting' algorithm which uses the transition matrix approach to solve the 'suboptimal' control class of problems. The shooting method is notorious for its sensitivity to the accuracy of the guesses of the initial Lagrange multipliers. Hull and Sheen have shown some remarkable results in terms of the accuracy of the Lagrange multipliers at the initial time with a relatively few number of control nodes which yield performance indices within a few percent of the optimal index.

This dissertation extends the work done in [10] by developing the necessary conditions and sufficient conditions for minima for optimal control problems with parameterized control. In addition, conditions are developed for the class of problems where there are both parameterized and nonparameterized controls. The need for such conditions may arise when solving certain

intercept problems in which the missile is initially moving out of the intercept plane and a bank needs to be commanded initially to get into the intercept plane. Another use of these conditions maybe for problems in which there is a switch time for control discontinuities or for the case of variable staging time for certain rocket launch problems. The second variation conditions for parameterized control are applied to a planar missile intercept problem.

Chapter 2 contains the derivation of the first variation necessary conditions for optimal control problems with parameters. A new variable, μ, is introduced, where the derivative of μ has the same form as the derivative of λ, which is the Lagrange multiplier associated with the state.

In Chapter 3, the neighboring extremal paths are derived using the sweep method. This results in a set of Riccati equations which need to be integrated backwards. This development involves taking the variation of the first variation necessary conditions derived in Chapter 2. The neighboring extremal parameterized 'control' law is developed directly from this and depends on the Riccati equations derived earlier. The neighboring extremal paths and parameterized control laws are developed.

Chapter 4 contains the development of the second variation, for the case of paramterized controls, which is obtained by taking the variation of the first variation. Then, a special perfect differential is added to (and subtracted from) the second variation and upon simplification yields a form whose integrand contains the form of the neighboring extremal control developed in Chapter 3. The second variation also has terms which are evaluated at the initial time. The conjugate point conditions are formulated as a result of this.

Chapter 5 contains the application of the second variation conditions to a missile intercept problem. The model used is an EMRAAT (Enhanced

Medium Range Air-to-Air Technology) class missile with a single thrust phase. The first variation conditions are used to obtain the optimal trajectories using a shooting method. Two fairly representative scenarios are presented along with the characteristics of these optimal intercept trajectories. The sufficient conditions obtained in Chapter 4 are applied to these trajectories and are shown to be minimal.

In Chapter 6, the first variation necessary conditions are obtained for the class of problems which have both parameterized and nonparameterized controls. The variable μ is defined in a similar manner as it is in Chapter 2. The first integral is also obtained.

Chapter 7 contains the development of neighboring extremal paths for the case of parameterized and nonparameterized controls using the sweep method. The development closely follows that in Chapter 3. The neighboring extremal parameterized and nonparameterized controls are obtained for these paths in terms of the Riccati equations which are obtained.

In Chapter 8, the second variation sufficient conditions for a minimum are obtained for the class where there are both parameterized and nonparameterized controls. The development of these conjugate point conditions follows closely with that of Chapter 4.

Finally, Chapter 9 contains some conclusions and summarizes the important developments of this research. Recommendations for additional follow-up research are also included.

Chapter 2

First Variation Necessary Conditions for Problems with Parameterized Control

2.1 Introduction

In this chapter the first variation necessary conditions are derived for optimal control problems with parameterized control. First, the first variation necessary conditions are derived. The first integral is obtained directly from these first variation necessary conditions. These conditions are used to find extrema and therefore do not indicate whether these solutions are minima, maxima, or saddle points.

2.2 The First Order Necessary Conditions

An optimal control problem can be stated as follows:

$$\text{minimize } J = \bar{\phi}(x_f, t_f) + \int_{t_0}^{t_f} \bar{L}(t, x, u)\, dt \tag{2.1}$$

subject to the differential constraints

$$\dot{x} = \bar{f}(t, x, u) \tag{2.2}$$

and the terminal constraints

$$\bar{\psi}(x_f, t_f) = 0 \tag{2.3}$$

with the initial conditions

$$t_0 = t_{0_s} = given \tag{2.4}$$

$$x_0 = x_{0_s} = given. \tag{2.5}$$

Any free final time problem can be transformed to a fixed final time problem, including the final time as part of the parameter set, by using the transformation

$$t = (t_f - t_0)\tau + t_0 \tag{2.6}$$

$$dt = (t_f - t_0)d\tau \tag{2.7}$$

where τ goes from 0 to 1. In addition, the control is parameterized over $\tau_i \leq \tau \leq \tau_{i+1}$ as

$$u = u_i + (u_{i+1} - u_i)\frac{(\tau - \tau_i)}{(\tau_{i+1} - \tau_i)}, \tag{2.8}$$

where τ_i depends on the number of control nodes chosen.

When these relations are used, Eqs. (2.1) and (2.2) become

$$\text{minimize } J = \tilde{\phi}(x_f, t_f) + \int_0^1 (t_f - t_0)\tilde{L}(\tau, x, u_i, t_f)\, d\tau \tag{2.9}$$

subject to the differential constraints

$$x' = (t_f - t_0)\tilde{f}(\tau, x, u_i, t_f). \tag{2.10}$$

Therefore, the optimal control problem is restated as follows:

$$\text{minimize } J = \phi(x_f, p) + \int_0^1 L(\tau, x, p)\, d\tau \tag{2.11}$$

subject to the differential constraints

$$x' = f(\tau, x, p) \tag{2.12}$$

and the terminal constraints

$$\psi(x_f, p) = 0 \tag{2.13}$$

with the initial conditions

$$x_0 = x_{0_s} = given \tag{2.14}$$

with p defined as

$$p \triangleq \begin{bmatrix} u_1 \\ \vdots \\ u_{k-1} \\ t_f \end{bmatrix}, \tag{2.15}$$

where $k-1$ is the number of parameterized control nodes. This formulation of the performance index will be used henceforth.

The constraints are adjoined to the performance index using Lagrange multipliers and it is expressed as

$$\bar{J} = G(x_f, p, \nu) + \int_0^1 \left[H(\tau, x, p, \lambda) - \lambda^T x'\right] d\tau \tag{2.16}$$

where

$$G(x_f, p, \nu) \triangleq \phi(x_f, p) + \nu^T \psi(x_f, p) \tag{2.17}$$

$$H(\tau, x, p, \lambda) \triangleq L(\tau, x, p) + \lambda^T f(\tau, x, p). \tag{2.18}$$

In order to obtain the first variation necessary conditions, the variation of Eq. (2.16) is taken, yielding

$$\begin{aligned} \delta\bar{J} &= G_{x_f}\delta x_f + G_p\delta p + G_\nu\delta\nu + \int_0^1 [H_x\delta x \\ &\quad + H_\lambda\delta\lambda + H_p\delta p - \delta\lambda^T x' - \lambda^T\delta x'\Big] d\tau. \end{aligned} \tag{2.19}$$

After the $\lambda^T\delta x'$ term is integrated by parts, the first variation becomes

$$\begin{aligned} \delta\bar{J} &= (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + G_\nu\delta\nu + \lambda_0^T\delta x_0 \\ &\quad + \int_0^1 \left[(H_x + \lambda^{T\prime})\delta x + H_p\delta p + (f^T - x^{T\prime})\delta\lambda\right] d\tau \end{aligned} \tag{2.20}$$

which is further simplified using Eqs. (2.12) and (2.13) (with $\delta x_0 = 0$) to

$$\delta\bar{J} = (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + \int_0^1 \left[(H_x + \lambda^{T\prime})\delta x + H_p\delta p\right] d\tau. \tag{2.21}$$

Now, the Lagrange multipliers, λ, are chosen such that the coefficients of the dependent variations (δx) vanish, that is

$$\lambda' = -H_x^T \tag{2.22}$$

$$\lambda_f = G_{x_f}^T. \tag{2.23}$$

The first variation now reduces to

$$\delta \bar{J} = G_p \delta p + \int_0^1 H_p \delta p \, d\tau. \tag{2.24}$$

Since p is a constant set of parameters, the variations δp are constant, they are removed from the integral to yield

$$\delta \bar{J} = (G_p + \int_0^1 H_p \, d\tau)\delta p. \tag{2.25}$$

The first variation must vanish because δp can be either positive or negative. Therefore, the coefficient of δp must vanish, that is,

$$G_p + \int_0^1 H_p \, d\tau = 0. \tag{2.26}$$

Now, a new variable, μ, is introduced, so that this condition can be rewritten as follows

$$\mu' = -H_p^T(\tau, x, p, \lambda) \tag{2.27}$$

with the boundary conditions

$$\mu_0 = 0 \tag{2.28}$$

$$\mu_f = G_p^T. \tag{2.29}$$

To summarize, the first variation necessary conditions for a minimum are

$$x' = f(\tau, x, p) \tag{2.30}$$

$$\lambda' = -H_x^T(\tau, x, p, \lambda) \tag{2.31}$$

$$\mu' = -H_p^T(\tau, x, p, \lambda) \tag{2.32}$$

with the boundary conditions

$$\begin{aligned}
\tau_0 &= 0, \;\; x_0 = x_{0_s} && (2.33)\\
\mu_o &= 0 && (2.34)\\
\tau_f &= 1 && (2.35)\\
\psi(x_f, p) &= 0 && (2.36)\\
\lambda_f &= G^T_{x_f}(x_f, p, \nu) && (2.37)\\
\mu_f &= G^T_p(x_f, p, \nu). && (2.38)
\end{aligned}$$

For a problem with i states, k parameters, and l terminal constraints, Eqs. (2.30)-(2.32) are $2i + k$ equations. From Eqs (2.36)-(2.38), $l + i + k$ final conditions are obtained which allow for the solution of the constant Lagrange multipliers ν (l unknowns), the parameters p, and the final state x_f. These equations can be used to solve for the remaining constants of integration. Since the control does not enter into the endpoint function G, the condition in Eq. (2.38) can be expressed as

$$\begin{aligned}
\mu_{u_{i_f}} &= 0 && (2.39)\\
\mu_{t_{f_f}} &= G_{t_f}. && (2.40)
\end{aligned}$$

2.3 The First Integral

The first integral can be calculated quite readily from

$$\frac{dH}{d\tau} = H_\tau + H_x x' + H_p p' + H_\lambda \lambda'. \tag{2.41}$$

Since

$$x' = f, \; H_\lambda = f^T, \; \lambda' = -H_x^T, \; p' = 0, \tag{2.42}$$

the derivative becomes

$$\frac{dH}{d\tau} = H_\tau. \tag{2.43}$$

As a result, if H does not contain τ explicitly (i.e. $H_\tau = 0$), the first integral becomes

$$H = \text{Constant}. \tag{2.44}$$

Chapter 3

Neighboring Extremal Paths for Problems with Parameterized Control

3.1 Introduction

Neighboring extremal paths are those which lie in the neighborhood of the extremal path and which satisfy the first variation conditions. The purpose of analyzing this is that the development of the second variation sufficient condition (Jacobi condition) follows from the analysis of these paths. In this chapter the neighboring extremals are developed and analyzed using the backward sweep method.

3.2 Neighboring Extremals for Problems with Parameterized Control

Given an extremal path, that is, one which satisfies the first variation necessary conditions derived in Chapter 2, an admissible comparison path is one which lies in the neighborhood of the extremal and satisfies all of the constraints.

Suppose there exists a neighboring extremal path from a point $x_{q_*} = x_q + \delta x_q$ at time t_q to a neighboring terminal constraint manifold $\psi_* = \psi + \delta\psi$. This is illustrated in Figure 3.1. Then, an admissible comparison path is formed by letting δx_q and $\delta\psi$ go to zero. If the subscript 1 denotes a neighboring extremal path, the following conditions must hold:

$$x_1{}' = f(\tau, x_1, p_1) \tag{3.1}$$

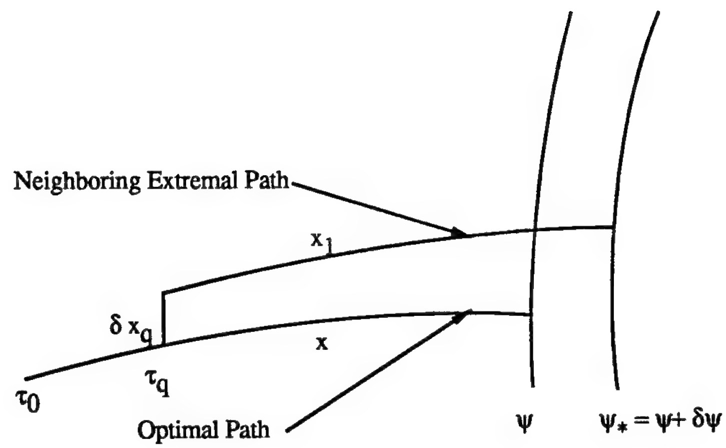

Figure 3.1: Neighboring Extremal Paths

$$\lambda_1{}' = -H_x^T(\tau, x_1, p_1, \lambda_1) \tag{3.2}$$

$$\mu_1{}' = -H_p^T(\tau, x_1, p_1, \lambda_1) \tag{3.3}$$

$$\psi(x_{1_f}, p_1) = 0,\ \lambda_{1_f} = G_{x_f}^T(x_{1_f}, p_1, \nu_1),\ \mu_{1_f} = G_p^T(x_{1_f}, p_1, \nu_1) \tag{3.4}$$

$$x_{1_0} = given,\ \mu_{1_0} = 0,\ \tau_0 = 0,\ \tau_f = 1 \tag{3.5}$$

Since $x_1(t)$ is a neighboring path to $x(t)$,

$$x_1 = x + \delta x,\quad \lambda_1 = \lambda + \delta\lambda,\quad \nu_1 = \nu + \delta\nu,\quad p_1 = p + \delta p. \tag{3.6}$$

If these equations are substituted into Eqs. (3.1)-(3.4), expanded in a Taylor series, and higher order terms (higher than one) are neglected, the resulting equations become

$$\delta x' = f_x \delta x + f_p \delta p \tag{3.7}$$

$$\delta\lambda' = -H_{xx}\delta x - f_x^T \delta\lambda - H_{xp}\delta p \tag{3.8}$$

$$\delta\mu' = -H_{px}\delta x - f_p^T \delta\lambda - H_{pp}\delta p \tag{3.9}$$

$$\delta\lambda_f = G_{x_f x_f}\delta x_f + \psi_{x_f}^T \delta\nu + G_{x_f p}\delta p \tag{3.10}$$

$$\delta\psi = \psi_{x_f}\delta x_f + \psi_p \delta p \tag{3.11}$$

$$\delta\mu_f = G_{p x_f}\delta x_f + \psi_p^T \delta\nu + G_{pp}\delta p. \tag{3.12}$$

In addition, the initial conditions are

$$\delta x_0 = given \tag{3.13}$$

$$\delta\mu_0 = given \tag{3.14}$$

In expressing Eqs. (3.8)-(3.12), the following relations have been used:

$$H_{x\lambda} = f_x^T, \quad G_{x_f\nu} = \psi_{x_f}^T, \quad G_{p\nu} = \psi_p^T. \tag{3.15}$$

Eqs. (3.7)-(3.9) can be rewritten as

$$\delta x' = \tilde{A}\delta x + \tilde{D}\delta p \tag{3.16}$$

$$\delta\lambda' = -\tilde{C}\delta x - \tilde{A}^T\delta\lambda - \tilde{E}\delta p \tag{3.17}$$

$$\delta\mu' = -\tilde{E}^T\delta x - \tilde{D}^T\delta\lambda - \tilde{F}\delta p \tag{3.18}$$

where

$$\tilde{A} = f_x \tag{3.19}$$

$$\tilde{C} = H_{xx} \tag{3.20}$$

$$\tilde{D} = f_p \tag{3.21}$$

$$\tilde{E} = H_{xp} \tag{3.22}$$

$$\tilde{F} = H_{pp}. \tag{3.23}$$

Also, Eqs. (3.10)-(3.12) can be expressed as

$$\delta\lambda_f = G_{x_f x_f}\delta x_f + \psi_{x_f}^T \delta\nu + G_{x_f p}\delta p \tag{3.24}$$

$$\delta\psi = \psi_{x_f}\delta x_f + 0\delta\nu + \psi_p \delta p \tag{3.25}$$

$$\delta\mu_f = G_{p x_f}\delta x_f + \psi_p^T \delta\nu + G_{pp}\delta p. \tag{3.26}$$

3.3 The Sweep Method

One way of solving the above linear two-point boundary-value problem is called the sweep method. This technique hypothesizes that the solution is of the form of the final conditions, i.e.

$$\delta\lambda = S(\tau)\delta x + R(\tau)\delta\nu + m(\tau)\delta p \tag{3.27}$$

$$\delta\psi = T(\tau)\delta x + Q(\tau)\delta\nu + n(\tau)\delta p \tag{3.28}$$

$$\delta\mu = \beta(\tau)\delta x + \gamma(\tau)\delta\nu + \alpha(\tau)\delta p \tag{3.29}$$

where

$$S_f = G_{x_f x_f} \tag{3.30}$$

$$R_f = \psi_{x_f}^T \tag{3.31}$$

$$m_f = G_{x_f p} \tag{3.32}$$

$$T_f = \psi_{x_f} \tag{3.33}$$

$$Q_f = 0 \tag{3.34}$$

$$n_f = \psi_p \tag{3.35}$$

$$\beta_f = G_{p x_f} \tag{3.36}$$

$$\gamma_f = \psi_p^T \tag{3.37}$$

$$\alpha_f = G_{pp}. \tag{3.38}$$

Differentiating Eq. (3.27) yields

$$\delta\lambda' = S'\delta x + S\delta x' + R'\delta\nu + m'\delta p. \tag{3.39}$$

Substituting for $\delta x'$ and $\delta\lambda'$ from Eqs. (3.16) and (3.17), respectively, yields,

$$-\tilde{C}\delta x - \tilde{A}^T\delta\lambda - \tilde{E}\delta p = S'\delta x + S(\tilde{A}\delta x + \tilde{D}\delta p) + R'\delta\nu + m'\delta p. \tag{3.40}$$

Substituting for $\delta\lambda$ from Eq. (3.27) and collecting terms produces

$$\begin{aligned} 0 &= (S' + \tilde{C} + S\tilde{A} + \tilde{A}^T S)\delta x + (R' + \tilde{A}^T R)\delta\nu \\ &\quad + (m' + \tilde{E} + S\tilde{D} + \tilde{A}^T m)\delta p. \end{aligned} \tag{3.41}$$

Since this equation is valid for all variations in x, ν, and p, S, R, and m are chosen such that each of the coefficients of the variations must be zero, that is

$$S' = -\tilde{A}^T S - S\tilde{A} - \tilde{C}, \quad S_f = G_{x_f x_f} \tag{3.42}$$

$$R' = -\tilde{A}^T R, \quad R_f = \psi_{x_f}^T \tag{3.43}$$

$$m' = -\tilde{A}^T m - S\tilde{D} - \tilde{E}, \quad m_f = G_{x_f p}. \tag{3.44}$$

Notice that since $\tilde{C}$ and $G_{x_f x_f}$ are symmetric and since the differential equation for S^T and the corresponding boundary condition are the same as for S, $S = S^T$, which means that S is symmetric.

Similarly, differentiating Eq. (3.28), with $\delta\psi' = 0$, gives

$$0 = T'\delta x + T\delta x' + Q'\delta\nu + n'\delta p. \tag{3.45}$$

Substituting for $\delta x'$, yields

$$0 = T'\delta x + T(\tilde{A}\delta x + \tilde{D}\delta p) + Q'\delta\nu + n'\delta p. \tag{3.46}$$

Accumulating the coefficients of the common variations produces

$$0 = (T' + T\tilde{A})\delta x + (Q')\delta\nu + (n' + T\tilde{D})\delta p. \tag{3.47}$$

As before, T, Q, and n are chosen such that the coefficients of the variations vanish. This results in the following set of equations with the associated

boundary conditions

$$
\begin{aligned}
T' &= -T\tilde{A}, \quad T_f = \psi_{x_f} &(3.48)\\
Q' &= 0, \qquad Q_f = 0 &(3.49)\\
n' &= -T\tilde{D}, \quad n_f = \psi_p. &(3.50)
\end{aligned}
$$

Finally, differentiating Eq. (3.29), yields,

$$\delta\mu' = \beta'\delta x + \beta\delta x' + \gamma'\delta\nu + \alpha'\delta p. \qquad (3.51)$$

Substituting for $\delta x'$ and $\delta\mu'$ from Eqs. (3.16) and (3.18), respectively, yields

$$-\tilde{E}^T\delta x - \tilde{D}^T\delta\lambda - \tilde{F}\delta p = \beta'\delta x + \beta(\tilde{A}\delta x + \tilde{D}\delta p) + \gamma'\delta\nu + \alpha'\delta p. \qquad (3.52)$$

Substituting for $\delta\lambda$ from Eq. (3.27) and collecting terms produces

$$
\begin{aligned}
0 &= (\beta' + \tilde{E}^T + \beta\tilde{A} + \tilde{D}^T S)\delta x + (\gamma' + \tilde{D}^T R)\delta\nu\\
&\quad + (\alpha' + \tilde{F} + \beta\tilde{D} + \tilde{D}^T m)\delta p. &(3.53)
\end{aligned}
$$

This yields the three equations

$$
\begin{aligned}
\beta' &= -\beta\tilde{A} - \tilde{D}^T S - \tilde{E}^T, \quad \beta_f = G_{px_f} &(3.54)\\
\gamma' &= -\tilde{D}^T R, \qquad \gamma_f = \psi_p^T &(3.55)\\
\alpha' &= -\tilde{D}^T m - \beta\tilde{D} - \tilde{F}, \quad \alpha_f = G_{pp}. &(3.56)
\end{aligned}
$$

Notice that the differential equations and boundary conditions for R^T and T, m^T and β, and n^T and γ, respectively, are the same; therefore, $T = R^T$, $\beta = m^T$, and $\gamma = n^T$.

To summarize, the differential equations for S, R, Q, m, n, and α and their respective boundary conditions are

$$S' = -\tilde{A}^T S - S\tilde{A} - \tilde{C}, \qquad S_f = G_{x_f x_f} \qquad (3.57)$$

$$
\begin{aligned}
R' &= -\tilde{A}^T R, & R_f &= \psi_{x_f}^T & (3.58)\\
m' &= -\tilde{A}^T m - S\tilde{D} - \tilde{E}, & m_f &= G_{x_f p} & (3.59)\\
n' &= -R^T \tilde{D}, & n_f &= \psi_p & (3.60)\\
\alpha' &= -\tilde{D}^T m - m^T \tilde{D} - \tilde{F}, & \alpha_f &= G_{pp} & (3.61)\\
Q &= 0. & & & (3.62)
\end{aligned}
$$

S has a dimension of $i \times i$, R has a dimension of $i \times l$, Q has a dimension of $l \times l$, m has a dimension of $i \times k$, n has a dimension of $l \times k$, α has a dimension of $k \times k$.

3.4 The Neighboring Extremal Parameterized Control

Eqs. (3.27)-(3.29) can be written in matrix form as

$$
\begin{bmatrix} \delta\lambda \\ \delta\psi \\ \delta\mu \end{bmatrix} = \begin{bmatrix} S(\tau) & R(\tau) & m(\tau) \\ R^T(\tau) & 0 & n(\tau) \\ m^T(\tau) & n^T(\tau) & \alpha(\tau) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\nu \\ \delta p \end{bmatrix}. \tag{3.63}
$$

The last two equations of the matrix can be written in the form

$$
\begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix} = U^T \delta x + V \begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix} \tag{3.64}
$$

where

$$
\begin{aligned}
U &\triangleq \begin{bmatrix} R(\tau) & m(\tau) \end{bmatrix}, & (3.65)\\
V &\triangleq \begin{bmatrix} 0 & n(\tau) \\ n^T(\tau) & \alpha(\tau) \end{bmatrix}. & (3.66)
\end{aligned}
$$

Therefore, if V^{-1} exists,

$$
\begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix} = -V^{-1} U^T \delta x + V^{-1} \begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{3.67}
$$

Eq. (3.27) can be rewritten as

$$\delta\lambda = S\delta x + U \begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix}. \tag{3.68}$$

Use of Eq. (3.67) results in the following equation:

$$\delta\lambda = (S - UV^{-1}U^T)\delta x + UV^{-1} \begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{3.69}$$

If δx_0 and $\delta\mu_0$ are given, this specifies $\delta\lambda_0$ and therefore δx, $\delta\lambda$ and $\delta\mu$ can be obtained as a function of τ. Since the matrix and the components of V^{-1} appear repeatedly in this analysis, it is useful to define and partition it as

$$V^{-1} \triangleq \begin{bmatrix} a & b \\ b^T & c \end{bmatrix} \tag{3.70}$$

where a is of the same dimension as $Q = 0$ $(l \times l)$, b is of the same dimension as n $(l \times k)$, and c is the same dimension as α $(k \times k)$. The matrix V^{-1} is symmetric because V is symmetric. In addition, $\bar{S}$ is defined as

$$\bar{S} \triangleq S - UV^{-1}U^T \tag{3.71}$$

and $\delta\Phi$ is defined as

$$\delta\Phi \triangleq \begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{3.72}$$

If the definitions in Eqs. (3.65) and (3.70) are used, the variations $\delta\nu$ and δp can be written as

$$\delta\nu = -(aR^T + bm^T)\delta x + b\delta\mu + a\delta\psi \tag{3.73}$$

$$\delta p = -(b^T R^T + cm^T)\delta x + c\delta\mu + b^T\delta\psi. \tag{3.74}$$

The quantities $Rb + mc$ and $Ra + mb^T$ appear consistently and are defined as

$$M \triangleq b^T R^T + cm^T = \begin{bmatrix} b^T & c \end{bmatrix} U^T \tag{3.75}$$

$$N \triangleq aR^T + bm^T = \begin{bmatrix} a & b \end{bmatrix} U^T. \tag{3.76}$$

Therefore, $\delta\nu$ and δp can be expressed as

$$\delta p = -M\delta x + c\delta\mu + b^T\delta\psi \tag{3.77}$$

$$\delta\nu = -N\delta x + b\delta\mu + a\delta\psi. \tag{3.78}$$

Eq. (3.77) is the neighboring extremal parameterized 'control' law for neighboring extremal trajectories.

If the definitions of $\bar{S}$, V^{-1}, M, and N are used, the differential equation for $\delta\mu$ is expressed as

$$\begin{aligned}\delta\mu' &= -\left(H_{px} + f_p\bar{S} - H_{pp}M\right)\delta x - \left(H_{pp}b^T + f_p^T N^T\right)\delta\psi \\ &\quad - \left(H_{pp}c + f_p^T M^T\right)\delta\mu \end{aligned} \tag{3.79}$$

or equivalently as

$$\begin{aligned}\delta\mu' &= -\left(\tilde{E}^T + \tilde{D}^T\bar{S} - \tilde{F}M\right)\delta x - \left(\tilde{F}b^T + \tilde{D}^T N^T\right)\delta\psi \\ &\quad - \left(\tilde{F}c + \tilde{D}^T M^T\right)\delta\mu. \end{aligned} \tag{3.80}$$

The differential equations for V^{-1}, $\bar{S}$, M, and N are developed in great detail in Appendix A.

Chapter 4

Sufficient Conditions for Problems with Parameterized Control

4.1 Introduction

In order to investigate whether a particular set of parameters is minimizing (or maximizing), the second variation must be investigated. The conditions which arise from this investigation can be used to determine whether a particular parameterized control set which satisfies the first variation necessary conditions (hence is, at most, an extremal set of parameters), will minimize the performance index. First, the second variation will be derived from the first variation. Next the sufficient conditions will be expressed in terms of the Jacobi (conjugate point) condition.

4.2 The Second Variation

Recall that the first variation was expressed in Eq. (2.20) as

$$\begin{aligned} \delta\bar{J} &= (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + G_\nu\delta\nu + \lambda_0^T\delta x_0 \\ &\quad + \int_0^1 \left[(H_x - \lambda^{T\prime})\delta x + H_p\delta p + (f^T - x^{T\prime})\delta\lambda\right] d\tau. \end{aligned} \tag{4.1}$$

The second variation is obtained by taking the variation of the first variation, that is

$$\begin{aligned} \delta^2\bar{J} &= \delta x_f^T G_{x_f x_f}\delta x_f + \delta x_f^T G_{x_f p}\delta p - \delta x_f^T\delta\lambda_f + \delta p^T G_{p x_f}\delta x_f \\ &\quad + \delta p^T G_{pp}\delta p + \delta\psi^T\delta\nu + (\delta x^T G_{x_f\nu} + \delta p^T G_{p\nu})\delta\nu \end{aligned}$$

$$
\begin{aligned}
&+\delta\lambda_0^T\delta x_0 + \lambda_0^T\delta^2 x_0 + \delta\nu^T\left(G_{\nu x_f}\delta x_f + G_{\nu p}\delta p\right)\\
&+\int_0^1 \delta x^T H_{xx}\delta x + \delta x^T H_{x\lambda}\delta\lambda + \delta x^T H_{xp}\delta p\\
&-\delta x^T\delta\lambda' + \delta p^T H_{px}\delta x + \delta p^T H_{p\lambda}\delta\lambda + \delta p^T H_{pp}\delta p\\
&+\delta\lambda^T(f_x\delta x + f_p\delta p - \delta x')\, d\tau. \qquad (4.2)
\end{aligned}
$$

Upon integration by parts of $\delta x^T\delta\lambda'$ and making use of Eqs. (3.7) and (3.15), as well as the constraints, Eqs. (3.11) and (3.27) (with $\delta^2 x_0 = 0$), the second variation is reduced to

$$
\begin{aligned}
\delta^2\bar{J} &= \begin{bmatrix}\delta x_f^T & \delta p^T\end{bmatrix}\begin{bmatrix}G_{x_f x_f} & G_{x_f p}\\ G_{p x_f} & G_{pp}\end{bmatrix}\begin{bmatrix}\delta x_f\\ \delta p\end{bmatrix}\\
&+\int_0^1 \begin{bmatrix}\delta x^T & \delta p^T\end{bmatrix}\begin{bmatrix}H_{xx} & H_{xp}\\ H_{px} & H_{pp}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta p\end{bmatrix} d\tau. \qquad (4.3)
\end{aligned}
$$

4.3 The Perfect Differential

Now, analyze a perfect differential of the form,

$$
\begin{aligned}
\tilde{P} &= \frac{d}{d\tau}\left[\delta x^T\bar{S}\delta x + \delta x^T U V^{-1}\delta\Phi + \delta\Phi^T V^{-1}U^T\delta x - \delta\Phi^T V^{-1}\delta\Phi\right] \qquad (4.4)\\
&= \frac{d}{d\tau}\Big[\delta x^T\bar{S}\delta x + \delta x^T M^T\delta\mu + \delta\mu^T M\delta x + \delta x^T N^T\delta\psi\\
&\quad +\delta\psi^T N\delta x - \delta\psi^T a\,\delta\psi - \delta\psi^T b\,\delta\mu - \delta\mu^T b^T\delta\psi - \delta\mu^T c\,\delta\mu\Big]. \qquad (4.5)
\end{aligned}
$$

Now, introduce

$$
\delta x' = f_x\delta x + f_p\delta p = \tilde{A}\delta x + \tilde{D}\delta p. \qquad (4.6)
$$

$\tilde{P}$ can be expanded to yield,

$$
\begin{aligned}
\tilde{P} &= \delta x^T\left(\tilde{A}^T\bar{S} + \bar{S}\tilde{A} + \bar{S}'\right)\delta x + \delta p^T\tilde{D}^T\bar{S}\delta x\\
&+\delta x^T\bar{S}\tilde{D}\delta p + \delta x^T\left(M^{T\prime} + \tilde{A}^T M^T\right)\delta\mu\\
&+\delta\mu^T\left(M' + M\tilde{A}\right)\delta x + \delta p^T\tilde{D}^T M^T\delta\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\delta\mu^T M\tilde{D}\delta p + \delta p^T \tilde{D}^T N^T \delta\psi \\
&+\delta\psi^T N\tilde{D}\delta p + \delta\psi^T \left(N' + N\tilde{A}\right)\delta x \\
&+\delta x^T \left(N^{T\prime} + \tilde{A}^T N^T\right)\delta\psi \\
&-\delta\mu^{T\prime}\left(-M\delta x + b^T\delta\psi + c\delta\mu\right) \\
&-\left(-\delta x^T M^T + \delta\psi^T b + \delta\mu^T c\right)\delta\mu' \\
&-\delta\psi^T a'\delta\psi - \delta\psi^T b'\delta\mu - \delta\mu^T b^{T\prime}\delta\psi \\
&-\delta\mu^T c'\delta\mu.
\end{aligned}
\tag{4.7}
$$

If certain terms are added and subtracted in the perfect differential, $\tilde{P}$ can be further expanded as follows

$$
\begin{aligned}
\tilde{P} \;=\; & \delta x^T \left[\tilde{A}^T\bar{S} + \bar{S}\tilde{A} + \bar{S}' + \tilde{C} - H_{xx} - \tilde{E}M - M^T\tilde{E}^T\right. \\
& \left.+M^T\tilde{F}M - \bar{S}\tilde{D}M - M^T\tilde{D}^T\bar{S} + M^T H_{pp}M\right]\delta x \\
& +\delta x^T\bar{S}f_p\delta p + \delta p^T f_p^T\bar{S}\delta x + \delta\mu^T M f_p\delta p + \delta p^T f_p^T M^T\delta\mu \\
& +\delta x^T\left[N^{T\prime} + \tilde{A}^T N^T + \bar{S}\tilde{D}b^T - M^T\tilde{D}^T N^T - M^T\tilde{F}b^T\right. \\
& \left.+\tilde{E}b^T - M^T H_{pp}b^T\right]\delta\psi + \delta\psi^T\left[N' + N\tilde{A} + b\tilde{D}^T\bar{S}\right. \\
& \left.+b\tilde{E}^T - N\tilde{D}M - b\tilde{F}M - bH_{pp}M\right]\delta x \\
& +\delta x^T\left[M^{T\prime} + \tilde{A}^T M^T + \tilde{E}c + \bar{S}\tilde{D}c - M^T\tilde{D}^T M^T\right. \\
& \left.-M^T\tilde{F}c - M^T H_{pp}c\right]\delta\mu + \delta\mu^T\left[M' + M\tilde{A}\right. \\
& \left.+c\tilde{D}^T\bar{S} - M\tilde{D}M - c\tilde{F}M + c\tilde{E}^T - cH_{pp}M\right]\delta x \\
& +\delta\psi^T\left[-b' + N\tilde{D}c + b\tilde{D}^T M^T + b\tilde{F}c + bH_{pp}c\right]\delta\mu \\
& +\delta\mu^T\left[-b^{T\prime} + c\tilde{D}^T N^T + M\tilde{D}b^T + c\tilde{F}b^T + cH_{pp}b^T\right]\delta\psi \\
& +\delta\psi^T\left[-a' + N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T + bH_{pp}b^T\right]\delta\psi \\
& +\delta\mu^T\left[-c' + M\tilde{D}c + c\tilde{D}^T M^T + cFc + cH_{pp}c\right]\delta\mu \\
& +\delta p^T f_p^T N^T\delta\psi - \left[\delta\mu^{T\prime} + \delta x^T\left(H_{xp} + \bar{S}f_p - M^T H_{pp}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\delta\psi^T \left(bH_{pp} + Nf_p\right) + \delta\mu^T \left(cH_{pp} + Mf_p\right)\Big] \\
& \cdot \left(-M\delta x + b^T\delta\psi + c\delta\mu\right) - \left(-\delta x^T M^T + \delta\psi^T b + \delta\mu^T c\right) \\
& \cdot \Big[\delta\mu' + \left(H_{px} + f_p^T \bar{S} - H_{pp}M\right)\delta x + \left(H_{pp}b^T + f_p^T N^T\right)\delta\psi \\
& + \left(H_{pp}c + f_p^T M^T\right)\delta\mu\Big] + \delta\psi^T N f_p \delta p
\end{aligned} \tag{4.8}
$$

Recognizing certain terms in $\tilde{P}$ as the neighboring extremal parameters as follows,

$$
\delta\bar{p} \;=\; -M\delta x + c\delta\mu + b^T\delta\psi, \tag{4.9}
$$

and adding and subtracting P and $\left(\delta\mu^{T\prime}\delta p + \delta p^T\delta\mu'\right)$ to the second variation defined in Eq. (4.3), $\delta^2\bar{J}$ becomes,

$$
\begin{aligned}
\delta^2\bar{J} \;=\; & \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\
& + \int_0^1 \Big\{ \left(\delta p - \delta\bar{p}\right)^T H_{pp} \left(\delta p - \delta\bar{p}\right) \\
& + \delta x^T \Big[\bar{S}' + \tilde{A}^T\bar{S} + \bar{S}\tilde{A} + \tilde{C} - \tilde{E}M - M^T\tilde{E}^T \\
& + M^T\tilde{F}M - \bar{S}\tilde{D}M - M^T\tilde{D}^T\bar{S}\Big]\delta x + \delta x^T \Big[N^{T\prime} \\
& + \tilde{A}^T N^T + \bar{S}\tilde{D}b^T - M^T\tilde{D}^T N^T - M^T\tilde{F}b^T + \tilde{E}b^T\Big]\delta\psi \\
& + \delta\psi \Big[N' + N\tilde{A} + b\tilde{D}^T\bar{S} - N\tilde{D}M - b\tilde{F}M + b\tilde{E}^T\Big]\delta x \\
& + \delta x^T \Big[M^{T\prime} + \tilde{A}^T M^T + \bar{S}\tilde{D}c - M^T\tilde{D}^T M^T - M^T\tilde{F}c + \tilde{E}c\Big]\delta\mu \\
& + \delta\mu^T \Big[M' + M\tilde{A} + c\tilde{D}^T\bar{S} - M\tilde{D}M - c\tilde{F}M \\
& + c\tilde{E}^T\Big] - \delta\psi^T \Big[a' - N\tilde{D}b^T - b\tilde{D}^T N^T - b\tilde{F}b^T\Big]\delta\psi \\
& - \delta\psi^T \Big[b' - N\tilde{D}c - b\tilde{D}^T M^T - b\tilde{F}c\Big]\delta\mu \\
& - \delta\mu^T \Big[b^{T\prime} - M\tilde{D}b^T - c\tilde{D}^T N^T - c\tilde{F}b^T\Big]\delta\psi \\
& - \delta\mu^T \Big[c' - M\tilde{D}c - c\tilde{D}^T M^T - c\tilde{F}c\Big]\delta\mu \\
& + \Big[\delta\mu^{T\prime} + \delta x^T\left(H_{xp} + \bar{S}f_p - M^T H_{pp}\right)
\end{aligned}
$$

$$+\delta\psi^T\left(bH_{pp}+Nf_p\right)+\delta\mu^T\left(cH_{pp}+Mf_p\right)\Big]\left(\delta p-\delta\bar{p}\right)$$
$$+\left(\delta p^T-\delta\bar{p}^T\right)\Big[\delta\mu'+\left(H_{px}+f_p^T\bar{S}-H_{pp}M\right)\delta x$$
$$+\left(H_{pp}b^T+f_p^TN^T\right)\delta\psi+\left(H_{pp}c+f_p^TM^T\right)\delta\mu\Big]$$
$$-\delta p^T\delta\mu'-\delta\mu^{T\prime}\delta p-\frac{d}{d\tau}\Big[\delta x^T\bar{S}\delta x+\delta x^TUV^{-1}\delta\Phi$$
$$\delta\Phi^TV^{-1}U^T\delta x-\delta\Phi^TV^{-1}\delta\Phi\Big]\Big\}\ d\tau. \qquad (4.10)$$

Now, $\bar{S}$, M, N, a, b, c, and $\delta\mu$ are chosen to satisfy the following relations, similar to the reasoning in Bryson and Ho[4],

$$\begin{aligned}
\bar{S}' &= -\tilde{A}^T\bar{S}-\bar{S}\tilde{A}-\tilde{C}+\tilde{E}M+M^T\tilde{E}^T \\
&\quad -M^T\tilde{F}M+\bar{S}\tilde{D}M+M^T\tilde{D}^T\bar{S} && (4.11)\\
M' &= -M\tilde{A}-c\tilde{D}^T\bar{S}+M\tilde{D}M+c\tilde{F}M-c\tilde{E}^T && (4.12)\\
N' &= -N\tilde{A}-b\tilde{D}^T\bar{S}+N\tilde{D}M+b\tilde{F}M-b\tilde{E}^T && (4.13)\\
a' &= N\tilde{D}b^T+b\tilde{D}^TN^T+b\tilde{F}b^T && (4.14)\\
b' &= N\tilde{D}c+b\tilde{D}^TM^T+b\tilde{F}c && (4.15)\\
c' &= M\tilde{D}c+c\tilde{D}^TM^T+c\tilde{F}c && (4.16)\\
\delta\mu' &= -\left(H_{px}+f_p^T\bar{S}-H_{pp}M\right)\delta x-\left(H_{pp}b^T+f_p^TN^T\right)\delta\psi \\
&\quad -\left(H_{pp}c+f_p^TM^T\right)\delta\mu && (4.17)
\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}
\bar{S}_f &= G_{x_fx_f}-\begin{bmatrix}\psi_{x_f}^T & G_{x_fp}\end{bmatrix}\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1}\begin{bmatrix}\psi_{x_f}\\ G_{px_f}\end{bmatrix} && (4.18)\\
\begin{bmatrix}N_f^T & M_f^T\end{bmatrix} &= U_fV_f^{-1}=\begin{bmatrix}\psi_{x_f}^T & G_{x_fp}\end{bmatrix}\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1} && (4.19)\\
V_f^{-1} &= \begin{bmatrix}a_f & b_f\\ b_f^T & c_f\end{bmatrix}=\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1} && (4.20)\\
\delta\mu_f &= G_{px_f}\delta x_f+\psi_p^T\delta\bar{\nu}+G_{pp}\delta\bar{p} && (4.21)\\
\delta\mu_0 &= given, && (4.22)
\end{aligned}$$

where the bar over p and ν denotes those quantities correspond to the neighboring extremal trajectory which terminates at the same location on the terminal constraint manifold as the admissible comparison path. Using these relations, the second variation becomes

$$\begin{aligned}\delta^2 \bar{J} &= \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\ &+ \int_0^1 \Big\{ (\delta p - \delta \bar{p})^T H_{pp} (\delta p - \delta \bar{p}) - \delta p^T \delta \mu' - \delta \mu^{T'} \delta p \\ &- \frac{d}{d\tau} \Big[\delta x^T \bar{S} \delta x + \delta x^T U V^{-1} \delta \Phi + \delta \Phi^T V^{-1} U^T \delta x \\ &- \delta \Phi^T V^{-1} \, \delta \Phi \Big] \Big\} \, d\tau. \end{aligned} \tag{4.23}$$

Recognizing that the second, third, and fourth terms can be integrated directly, the second variation can be further reduced to

$$\begin{aligned}\delta^2 \bar{J} &= \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\ &- \Big[\delta p^T \delta \mu + \delta \mu^T \delta p \Big]_0^1 - \Big[\delta x^T \bar{S} \delta x + \delta x^T U V^{-1} \delta \Phi \\ &+ \delta \Phi^T V^{-1} U^T \delta x - \delta \Phi^T V^{-1} \, \delta \Phi \Big]_0^1 \\ &+ \int_0^1 (\delta p - \delta \bar{p})^T H_{pp} (\delta p - \delta \bar{p}) \; d\tau. \end{aligned} \tag{4.24}$$

The terms outside the integral are separated into those at the initial time and those at the final time as follows:

$$\delta^2 \bar{J} = Y_o + Y_f + \int_0^1 (\delta p - \delta \bar{p})^T H_{pp} (\delta p - \delta \bar{p}) \; d\tau \tag{4.25}$$

where Y_o and Y_f are defined by

$$\begin{aligned} Y_o &= \delta p^T \delta \mu_o + \delta \mu_o^T \delta p + \delta x_o^T \bar{S}_o \delta x_o + \delta x_o^T U_o V_o^{-1} \delta \Phi_o \\ &+ \delta \Phi_o^T V_o^{-1} U_o^T \delta x_o - \delta \Phi_o^T V_o^{-1} \delta \Phi_o \end{aligned} \tag{4.26}$$

$$\begin{aligned} Y_f &= \delta x_f^T G_{x_f x_f} \delta x_f + \delta x_f^T G_{x_f p} \delta p + \delta p^T G_{p x_f} \delta x_f + \delta p^T G_{pp} \delta p \\ &- \delta p^T \delta \mu_f - \delta \mu_f^T \delta p - \delta x_f^T \bar{S}_f \delta x_f - \delta x_f^T U_f V_f^{-1} \delta \Phi_f \\ &- \delta \Phi_f^T V_f^{-1} U_f^T \delta x_f + \delta \Phi_f^T V_f^{-1} \delta \Phi_f. \end{aligned} \tag{4.27}$$

After substituting for μ_f, $\bar{S}_f$, $U_f V_f^{-1}$, and V_f^{-1}, Eq. (4.27) is rewritten as

$$\begin{aligned} Y_f &= \delta p^T G_{pp} \delta p - \delta p^T \left[\psi_p^T \delta \bar{\nu} + G_{pp} \delta \bar{p} \right] - \left[\delta \bar{\nu}^T \psi_p + \delta \bar{p} G_{pp} \right] \delta p \\ &+ \delta x_f^T \left\{ \begin{bmatrix} \psi_{x_f}^T & G_{x_f p} \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \psi_{x_f} \\ G_{px_f} \end{bmatrix} \right\} \delta x_f \\ &- \delta x_f^T \begin{bmatrix} \psi_{x_f}^T & G_{x_f p} \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix} \\ &- \begin{bmatrix} \delta \psi^T & \delta \mu_f^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \psi_{x_f} \\ G_{px_f} \end{bmatrix} \delta x_f \\ &+ \begin{bmatrix} \delta \psi^T & \delta \mu_f^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix}. \end{aligned} \tag{4.28}$$

Recall that on the neighboring extremal path, Eq. (3.64) evaluated at the final time is expressed as

$$\begin{bmatrix} \psi_{x_f} \\ G_{px_f} \end{bmatrix} \delta x_f = - \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta \bar{\nu} \\ \delta \bar{p} \end{bmatrix} + \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix} \tag{4.29}$$

Eq. (4.29) is used to reduce Eq. (4.28) to

$$\begin{aligned} Y_f &= \delta p^T G_{pp} \delta p - \delta p^T G_{pp} \delta \bar{p} - \delta \bar{p}^T G_{pp} \delta p - \delta p^T \psi_p^T \delta \bar{\nu} - \delta \bar{\nu}^T \psi_p \delta p \\ &+ \left\{ - \begin{bmatrix} \delta \bar{\nu}^T & \delta \bar{p}^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} + \begin{bmatrix} \delta \psi^T & \delta \mu_f^T \end{bmatrix} \right\} \cdot \\ &\cdot \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \left\{ - \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta \bar{\nu} \\ \delta \bar{p} \end{bmatrix} + \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix} \right\} \\ &- \left\{ - \begin{bmatrix} \delta \bar{\nu}^T & \delta \bar{p}^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} + \begin{bmatrix} \delta \psi^T & \delta \mu_f^T \end{bmatrix} \right\} \\ &\cdot \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta \psi^T \\ \delta \mu_f^T \end{bmatrix} - \begin{bmatrix} \delta \psi & \delta \mu_f \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \cdot \\ &\cdot \left\{ - \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta \bar{\nu} \\ \delta \bar{p} \end{bmatrix} + \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix} \right\} \\ &+ \begin{bmatrix} \delta \psi^T & \delta \mu_f^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta \psi \\ \delta \mu_f \end{bmatrix} \end{aligned} \tag{4.30}$$

which simplifies to yield

$$Y_f = \delta p^T G_{pp} \delta p - \delta p^T G_{pp} \delta \bar{p} - \delta \bar{p}^T G_{pp} \delta p - \delta p^T \psi_p^T \delta \bar{\nu}$$

$$-\delta\bar{\nu}^T\psi_p\delta p + \begin{bmatrix} \delta\bar{\nu}^T & \delta\bar{p}^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta\bar{\nu} \\ \delta\bar{p} \end{bmatrix} \tag{4.31}$$

which further reduces to

$$\begin{aligned} Y_f &= (\delta p - \delta\bar{p})^T G_{pp} (\delta p - \delta\bar{p}) - (\delta p - \delta\bar{p})^T \psi_p^T \delta\bar{\nu} \\ &\quad -\delta\bar{\nu}^T \psi_p (\delta p - \delta\bar{p}) . \end{aligned} \tag{4.32}$$

At the terminal constraint manifold, the neighboring extremal and the admissible comparison path, respectively, must satisfy the following equations

$$0 = \psi_{x_f}\delta x_f + \psi_p\delta\bar{p} \tag{4.33}$$

$$0 = \psi_{x_f}\delta x_f + \psi_p\delta p, \tag{4.34}$$

Therefore, if these two equations are subtracted, the resulting equation is

$$\psi_p (\delta p - \delta\bar{p}) = 0 \tag{4.35}$$

With this relation, Y_f can be finally reduced to

$$Y_f = (\delta p - \delta\bar{p})^T G_{pp} (\delta p - \delta\bar{p}) . \tag{4.36}$$

Similarly, Y_o, from Eq. (4.26), is written as

$$\begin{aligned} Y_o &= \delta p^T\delta\mu_o + \delta\mu_o^T\delta p + \delta x_o^T\bar{S}\delta x_o + \delta x_o^T U_o V_o^{-1}\delta\Phi \\ &\quad +\delta\Phi^T V_o^{-1} U_o^T\delta x_o - \delta\Phi^T V_o^{-1}\delta\Phi \end{aligned} \tag{4.37}$$

$$\begin{aligned} &= \delta p^T\delta\mu_o + \delta\mu_o^T\delta p + \delta x_o^T\bar{S}\delta x_o + \delta x_o^T N_o^T\delta\psi \\ &\quad \delta\psi^T N_o\delta x_o + \delta x_o^T M_o^T\delta\mu_o + \delta\mu_o^T M_o\delta x_o - \delta\psi^T a_o\delta\psi \\ &\quad -\delta\mu_o^T b_o\delta\psi - \delta\psi^T b_o^T\delta\mu_o - \delta\mu_o^T c_o\delta\mu_o \end{aligned} \tag{4.38}$$

Since $\delta\mu_0 = 0$ and $\delta\psi = 0$, Y_o becomes

$$Y_o = \delta x_o^T\bar{S}_o\delta x_o \tag{4.39}$$

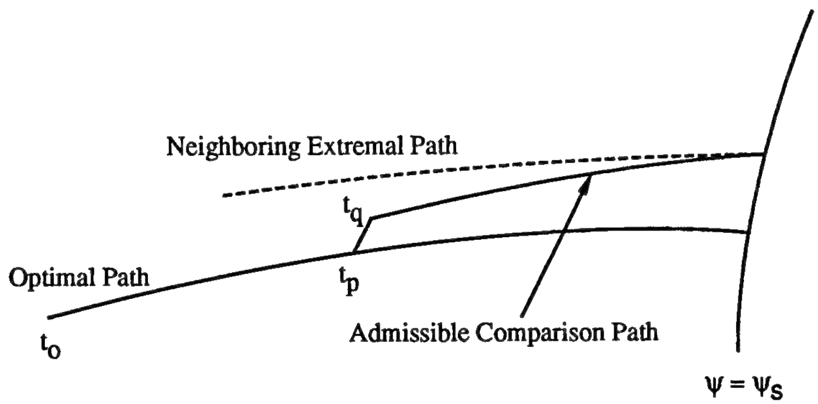

Figure 4.1: Admissible Comparison Paths

Therefore, using Eqs. (4.36) and (4.39), the second variation in Eq. (4.25) becomes

$$\delta^2 \bar{J} \;=\; \delta x_o^T \bar{S}_o \delta x_o + (\delta p - \delta \bar{p})^T \left[G_{pp} + \int_0^1 H_{pp}\, d\tau \right] (\delta p - \delta \bar{p})\,. \qquad (4.40)$$

For an admissible comparison path, that is one which lies in the neighborhood of the extremal and which satisfies all the constraints, $\delta x_o = 0$, $\delta\psi = 0$, and $\delta\mu_o = 0$. Therefore, if $\bar{S}_o$ and $U_o V_o^{-1}$ are finite over $[0,1)$, the initial point term outside the integral in Eq. (4.40) vanishes and the second variation reduces to

$$\delta^2 \bar{J} = (\delta p - \delta \bar{p})^T \left[G_{pp} + \int_0^1 H_{pp}\, d\tau \right] (\delta p - \delta \bar{p})\,. \qquad (4.41)$$

Hence, if $G_{pp} + \int_0^1 H_{pp}\, d\tau > 0$, the second variation is positive for arbitrary variations δp, unless $\delta p = \delta \bar{p}$. This can occur only if the admissible comparison path is a neighboring extremal. However, Eq. (3.69) indicates that for $\bar{S}_o$ and $U_o V_o^{-1}$ finite and $\delta x_o = 0$, $\delta \mu_o = 0$, and $\delta\psi = 0$, this results in $\delta\lambda_o = 0$.

With this, Eq. (3.77) reduces to $\delta p = 0$. Therefore, Eqs. (3.16) - (3.18) simplify to $\delta x = \delta\lambda = \delta\mu = 0$, which implies that $\delta p = 0$, or that there is no admissible comparison path which is a neighboring extremal path. Therefore, the extremal path is a minimum.

On the other hand, if $\bar{S}_o$ and $U_o V_o^{-1}$ become infinite, $\delta x_o = 0$, $\delta\mu_o = 0$, and $\delta\psi = 0$ could lead to a finite $\delta\lambda_o$ and an admissible comparison path which is a neighboring extremal. Therefore, the second variation could become negative and the extremal path is not a minimum. This is also the case when $\bar{S}$, UV^{-1}, and V^{-1} become infinite at some point within the interval. If this happens, this point is called a conjugate point.

The sufficient conditions are formulated from the operations needed to form $\delta^2\bar{J}$ as a perfect square. In addition, since a perfect differential is added to the second variation, $\bar{S}$, UV^{-1}, and V^{-1} must be finite for a finite δx and $\delta\Phi$ to lead to a finite $\delta\nu$ and δp, as well as to a finite $\delta\lambda$. These conditions can also be stated as requiring that the derivatives of $\bar{S}$, UV^{-1}, and V^{-1} be integrable or $\bar{S}$, UV^{-1}, and V^{-1} be finite over $[0,1)$. The following theorem states the sufficient conditions.

Theorem 1 *If the first variation necessary conditions for a minimum are satisfied and the following conditions are satisfied,*

1) $G_{pp} + \int_0^1 H_{pp}\, d\tau > 0$,

2) $\bar{S}$ *finite over* $0 \le \tau < 1$,

3) UV^{-1} *finite over* $0 \le \tau < 1$, *and*

4) V^{-1} *finite over* $0 \le \tau < 1$,

then this solution is a minimum.

A special case of this theorem arises when t_f only appears linearly in the variational Hamiltonian, H, and the endpoint function, G. In this case, $H_{t_f t_f}$ and $G_{t_f t_f}$ are identically zero. Therefore,

$$H_{t_f u_p} = \frac{H_{u_p}}{t_f}, \tag{4.42}$$

where u_p refers to the set of parameterized controls which excludes t_f.

The first condition in Theorem 1 becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \int_0^1 \begin{bmatrix} H_{u_p u_p} & \frac{H_{u_p}^T}{t_f} \\ \frac{H_{u_p}}{t_f} & 0 \end{bmatrix} d\tau \geq 0. \tag{4.43}$$

However, from the first variation,

$$\int_0^1 H_{u_p}\, d\tau = 0. \tag{4.44}$$

Therefore, Condition 1 in Theorem 1 can be reduced to

$$\int_0^1 H_{u_p u_p} d\tau > 0. \tag{4.45}$$

Chapter 5

Application of the Sufficient Conditions to a Missile Intercept Problem

In this chapter the sufficient conditions for a minimum, obtained in Chapter 4, are applied to a missile intercept problem. First, the missile model, which includes the thrust and aerodynamic characteristics, is presented. The optimization problem is described and the resulting first variation conditions are presented. Two realistic missile engagement scenarios are presented. The first case is for a target toward the launching aircraft and the second is for a target moving away from the launching aircraft. The matrices required in the application of the second variation conditions are presented and the sufficient conditions for a minimum are applied to this solution to determine whether it is indeed a minimum.

5.1 The Missile Model

The model used to verify the sufficient conditions is a EMRAAT (Extended Medium Range Air-to-Air Technology) -class missile. The thrust and aerodynamic parameters are chosen to reflect the performance characteristics of this type of missile. It has a single thrust phase which produces 4000 lb and has a duration of 9.47 seconds. The drag parameters are constant and are expressed as a parabolic drag polar of the following form

$$C_D = C_{D_0} + KC_L^2 \tag{5.1}$$

where

$$C_{D_0} = 0.4453 \tag{5.2}$$

$$K = 0.0171. \tag{5.3}$$

At launch, the missile weighs 375 lb and at engine burnout the weight reduces to 227 lb. This class of missiles was designed to be used for ranges between 20 and 40 miles at altitudes ranging from 10,000 ft to 50,000 ft. The atmosphere chosen is an exponential atmosphere of the following form

$$\rho = \rho_0 e^{-h/h_s} \tag{5.4}$$

where

$$\rho_0 = 0.0023769 \quad \text{slugs/ft}^3 \tag{5.5}$$

$$h_s = 23,800 \quad \text{ft.} \tag{5.6}$$

5.2 Equations of Motion

The two-dimensional translational equations of motion are used to compute the trajectory of the missile. The reason for this is that it was observed that even when the full three-dimensional equations were used, the missile immediately turned into the plane of the intercept. The equations of motion in the vertical plane are as follows

$$\dot{x} = V\cos\gamma \tag{5.7}$$

$$\dot{h} = V\sin\gamma \tag{5.8}$$

$$\dot{V} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \tag{5.9}$$

$$\dot{\gamma} = \frac{L}{mV} - \frac{g\cos\gamma}{V} \tag{5.10}$$

where D and L are

$$D = \frac{1}{2}\rho V^2 S C_D \tag{5.11}$$

$$L = \frac{1}{2}\rho V^2 S C_L \tag{5.12}$$

$$S = 0.3068 \quad \text{ft}^2. \tag{5.13}$$

The control is chosen to be the coefficient of lift, C_L.

5.3 The Optimal Control Problem

The optimal control problem can be stated as follows: Find the coefficient of lift which maximizes the terminal energy at target intercept. Mathematically, it is expressed as follows

$$\text{maximize} \quad J = \frac{V_f^2}{2g} \tag{5.14}$$

subject to the dynamics

$$\dot{x} = V\cos\gamma \tag{5.15}$$

$$\dot{h} = V\sin\gamma \tag{5.16}$$

$$\dot{V} = \frac{T}{m} - \frac{\rho V^2 S C_D}{2m} - g\sin\gamma \tag{5.17}$$

$$\dot{\gamma} = \frac{\rho V S C_L}{2m} - \frac{g\cos\gamma}{V} \tag{5.18}$$

along with the terminal constraints

$$x_0 = 0 \tag{5.19}$$

$$h_0 = h_{0_s} = given \tag{5.20}$$

$$V_0 = V_{0_s} = given \tag{5.21}$$

$$\gamma_0 = \gamma_{0_s} = given \tag{5.22}$$

$$x_f = (V_T\cos\gamma_T)\,t_f + x_{T_0} \tag{5.23}$$

$$h_f = (V_T\sin\gamma_T)\,t_f + h_{T_0} \tag{5.24}$$

with the thrust profile, aerodynamics, and density as

$$T = 4000 \text{ lb} \quad \text{for} \quad t \leq 9.47 \text{ seconds} \tag{5.25}$$

$$= 0 \quad \text{for} \quad t > 9.47 \text{ seconds} \tag{5.26}$$

$$C_D = C_{D_0} + KC_L^2 \tag{5.27}$$

$$\rho = \rho_0 e^{-h/h_s}. \tag{5.28}$$

Sub-optimal control involves finding the optimal control in the class of piecewise linear controls which minimize the performance index. Therefore, the coefficient of lift, C_L, is parameterized into nine nodes, four in the thrust phase and five in the coast phase.

In addition, since the final time is not specified, it is included in the parameter set which is optimized. This is done by transforming the problem into a fixed final time problem using the transformation described in Eq.(2.6). The dynamical equations become

$$x' = t_f V \cos\gamma \tag{5.29}$$

$$h' = t_f V \sin\gamma \tag{5.30}$$

$$V' = t_f\left(\frac{T}{m} - \frac{\rho V^2 S C_D}{2m} - g\sin\gamma\right) \tag{5.31}$$

$$\gamma' = t_f\left(\frac{\rho V S C_L}{2m} - \frac{g\cos\gamma}{V}\right). \tag{5.32}$$

The Hamiltonian and the end-point function are

$$\begin{aligned} H &= \lambda_x t_f V\cos\gamma + \lambda_h t_f V \sin\gamma + \lambda_V t_f\left(\frac{T}{m} - \frac{\rho V^2 S C_D}{2m} - g\sin\gamma\right) \\ &\quad + \lambda_\gamma t_f\left(\frac{\rho V S C_L}{2m} - \frac{g\cos\gamma}{V}\right) \end{aligned} \tag{5.33}$$

$$\begin{aligned} G &= -\frac{V_f^2}{2g} + \nu_x\left[x_f - (V_T\cos\gamma_T)\,t_f - x_{T_0}\right] \\ &\quad + \nu_h\left[h_f - (V_T\sin\gamma_T)\,t_f - h_{T_0}\right]. \end{aligned} \tag{5.34}$$

The coefficient of lift is obtained from the relation

$$C_L = C_{L_i} + \left(C_{L_{i+1}} - C_{L_i}\right)\frac{(\tau - \tau_i)}{(\tau_{i+1} - \tau_i)}. \tag{5.35}$$

The Euler-Lagrange equations are applied to this problem and yield

$$\begin{aligned}
\lambda_x{}' &= 0 && (5.36)\\
\lambda_h{}' &= -\frac{\lambda_V t_f \rho V^2 S C_D}{2mh_s} + \frac{\lambda_\gamma t_f \rho V S C_L}{2mh_s} && (5.37)\\
\lambda_V{}' &= -\lambda_x t_f \cos\gamma - \lambda_h t_f \sin\gamma + \frac{\lambda_V t_f \rho V S C_D}{m} \\
&\quad -\frac{\lambda_\gamma t_f \rho S C_L}{2m} - \frac{\lambda_\gamma t_f g \cos\gamma}{V^2} && (5.38)\\
\lambda_\gamma{}' &= \lambda_x t_f V \sin\gamma - \lambda_h t_f V \cos\gamma + \lambda_V t_f g \cos\gamma - \frac{\lambda_\gamma t_f g \sin\gamma}{V}. && (5.39)
\end{aligned}$$

The boundary conditions associated with these multipliers at the terminal time are

$$\begin{aligned}
\lambda_{x_f} &= \nu_x && (5.40)\\
\lambda_{h_f} &= \nu_h && (5.41)\\
\lambda_{V_f} &= -\frac{V_f}{g} && (5.42)\\
\lambda_{\gamma_f} &= 0. && (5.43)
\end{aligned}$$

The Euler-Lagrange equations corresponding to the parameters are

$$\begin{aligned}
\mu_{t_f}{}' &= -\lambda_x V \cos\gamma - \lambda_h V \sin\gamma - \lambda_V\left(\frac{T}{m} - \frac{\rho V^2 S C_D}{2m} - g\sin\gamma\right) \\
&\quad -\lambda_\gamma\left(\frac{\rho V S C_L}{2m} - \frac{g\cos\gamma}{V}\right) && (5.44)\\
\mu_{C_L}{}' &= \frac{\lambda_V t_f \rho V^2 S K C_L}{m}\frac{\partial C_L}{\partial C_{L_i}} - \frac{\lambda_\gamma t_f \rho V S}{2m}\frac{\partial C_L}{\partial C_{L_i}} && (5.45)
\end{aligned}$$

with boundary conditions

$$\mu_{t_{f_0}} = 0 \tag{5.46}$$

$$\mu_{t_{ff}} = 0 \tag{5.47}$$

$$\mu_{C_{L0}} = 0 \tag{5.48}$$

$$\mu_{C_{Lf}} = 0. \tag{5.49}$$

5.4 Solution of the Optimal Control Problem

The equations described in the previous section have been implemented in a trajectory simulation. As stated earlier, nine control nodes were chosen, four in the first (thrust) interval, and five in the second (coast) phase. The optimal trajectory was obtained using a transition matrix based shooting method. This technique was first proposed in [8] and gives the optimal trajectory for parameterized control problems relatively easily. The advantage of this method is that the dynamics are decoupled from the guesses of the initial Lagrange multipliers. The penalty paid for this decoupling is that the solution obtained is not the true optimal. However, Hull and Sheen have demonstrated that the sub-optimal solutions obtained are within a percent or two of the true optimal. Since these techniques would be used on board aerospace vehicles, this tradeoff, between computation time, effort, and accuracy, seems to favor implementation of this suboptimal shooting algorithm. A complete description of this algorithm is presented in Appendix C. In addition, a sub-optimal trajectory simulation is developed using a non-linear recursive quadratic programming algorithm called VF02AD. The results from this algorithm for the optimal trajectory are compared with that obtained from the shooting method. Because of the nature of the algorithm, extensive scaling of the constraints and the performance index is done to ensure convergence. The convergence criterion for both sets of algorithms is chosen to be 10^{-8}.

Numerous trajectory simulations have been carried out for different

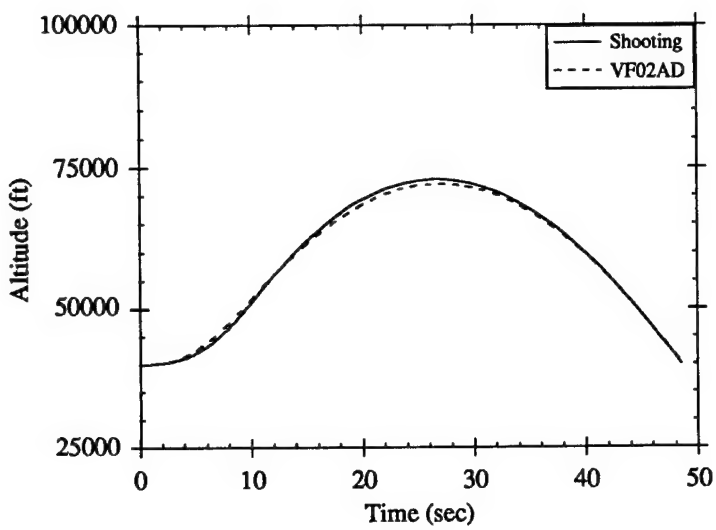

Figure 5.1: Altitude vs Time, Case 1

launch and target ranges and altitudes. The two most realistic cases are presented to demonstrate the usefulness and applicability of this technique.

In both cases, both the launching aircraft and the target are at 40,000 ft. In the first case, the target is initially at 40 miles moving toward the launching aircraft at 1,000 ft/sec. In the second case, the target is initially at 30 miles, moving away from the target at 1,000 ft/sec.

5.4.1 Case 1: Initial Distance 40 miles, target closing at 1,000 ft/sec

The results for this case are presented graphically. The plots include histories of altitude, velocity, flight path angle, and the coefficient of lift.

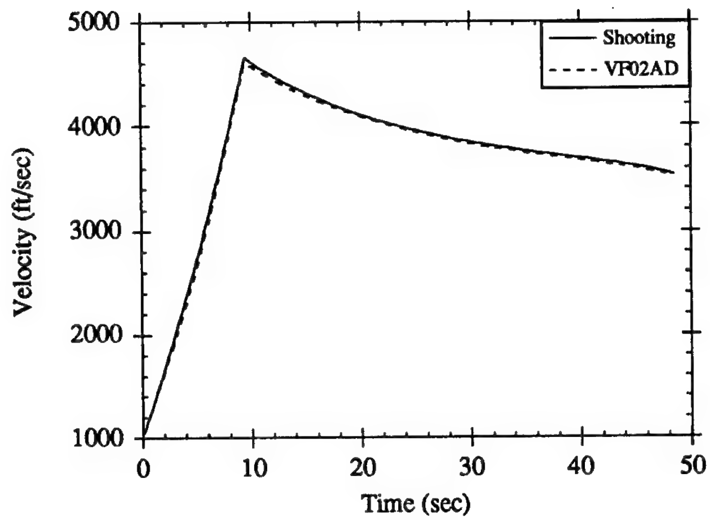

Figure 5.2: Velocity vs Time, Case 1

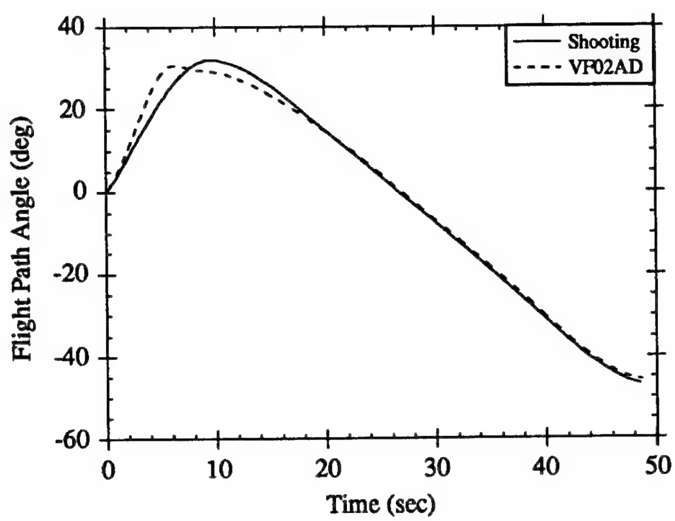

Figure 5.3: Flight Path Angle vs Time, Case 1

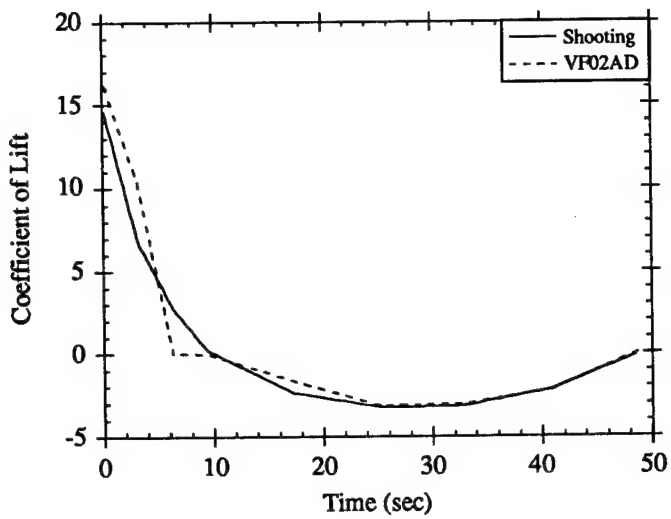

Figure 5.4: Coefficient of Lift vs Time, Case 1

5.4.2 Case 2: Initial Distance 30 miles, target moving away at 1,000 ft/sec

The results for this case are presented below. As in the previous case, the histories of altitude, velocity, flight path angle, and coefficient of lift are presented.

The plots show that the shooting method as well as VF02AD give similar results for the optimal trajectories. The altitude and velocity profiles are almost identical for the two techniques. The coefficient of lift histories for the two techniques are a bit different owing to the way VF02AD handles scaling when the controls are close to zero. The shooting method gives a reasonably smooth control history. As expected, if the number of control nodes is doubled, the control history is more smooth. The first case, in which the target was moving toward the missile, has a flight time of about 48 seconds;

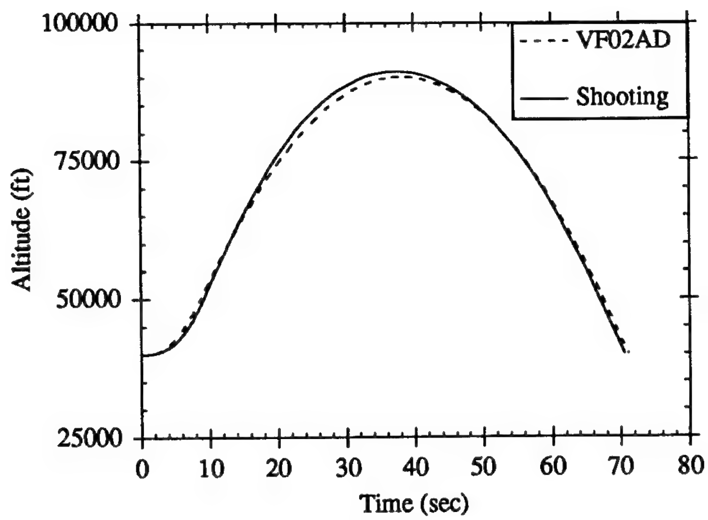

Figure 5.5: Altitude vs Time, Case 2

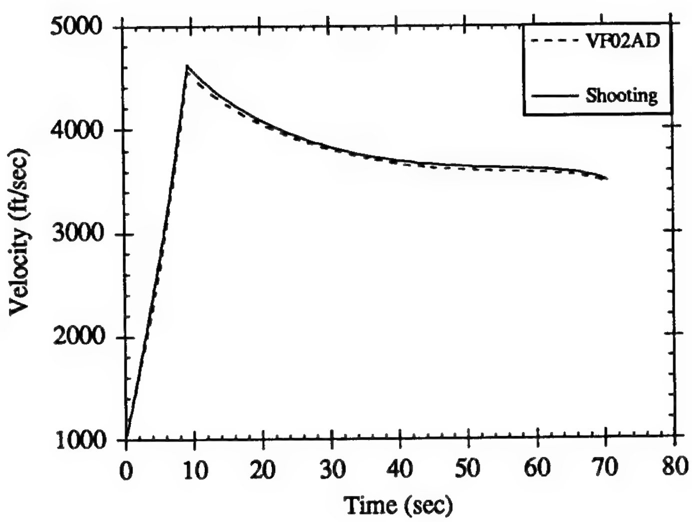

Figure 5.6: Velocity vs Time, Case 2

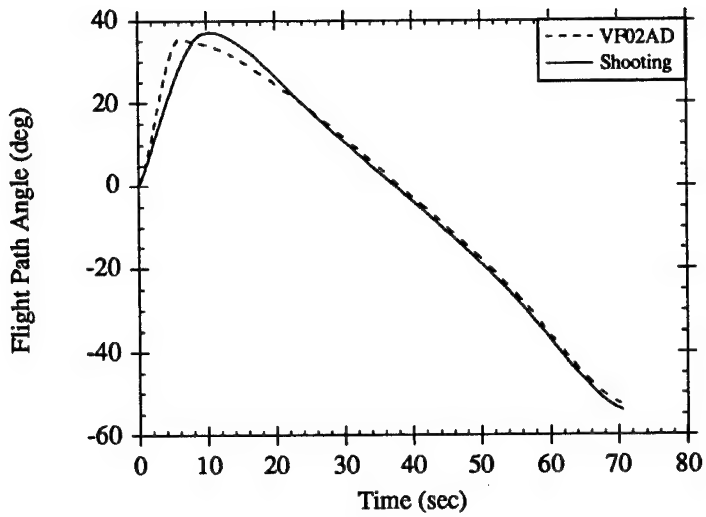

Figure 5.7: Flight Path Angle vs Time, Case 2

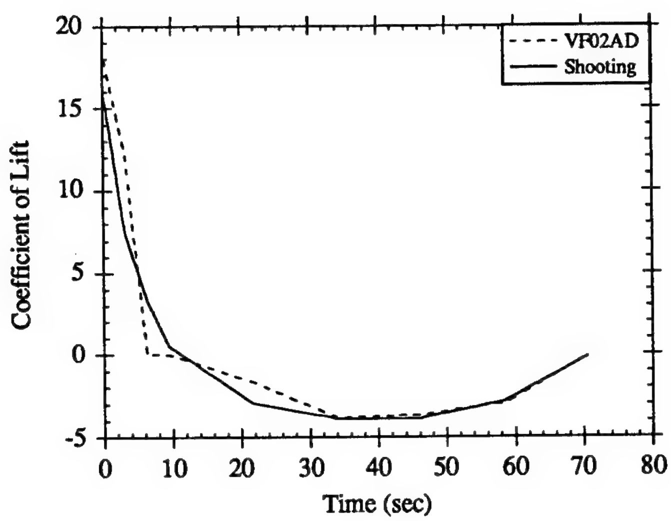

Figure 5.8: Coefficient of Lift vs Time, Case 2

in addition the missile lofts to about 73,000 ft. In the second case, in which the target is moving away from the missile, the flight time is closer to 72 seconds with a maximum altitude of 92,000 ft. In this case the missile travels farther to intercept the target, so it lofts higher to maximize the energy at intercept. These results seem consistent and are what was expected.

5.5 The Riccati Equations

The Riccati equations are integrated for the cases presented in the previous section. The matrices $\tilde{A}$-$\tilde{F}$ which appear as part of the equations are defined as follows

$$\tilde{A} = f_x = t_f \begin{bmatrix} 0 & 0 & \cos\gamma & -V\sin\gamma \\ 0 & 0 & \sin\gamma & V\cos\gamma \\ 0 & \frac{\rho V^2 S C_D}{2mh_s} & -\frac{\rho V S C_D}{m} & -g\cos\gamma \\ 0 & -\frac{\rho V S C_L}{2mh_s} & \left(\frac{\rho S C_L}{2m} + \frac{g\cos\gamma}{V^2}\right) & \frac{g\sin\gamma}{V} \end{bmatrix} \tag{5.50}$$

$$\tilde{C} = H_{xx} = t_f \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_2 & C_3 & C_4 \\ 0 & 0 & C_4 & C_5 \end{bmatrix} \tag{5.51}$$

where

$$C_1 = -\lambda_V\left(\frac{\rho V^2 S C_D}{2mh_s^2}\right) + \lambda_\gamma\left(\frac{\rho V S C_L}{2mh_s^2}\right) \tag{5.52}$$

$$C_2 = \lambda_V\left(\frac{\rho V S C_D}{mh_s}\right) - \lambda_\gamma\left(\frac{\rho S C_L}{2mh_s}\right) \tag{5.53}$$

$$C_3 = -\lambda_V\left(\frac{\rho S C_D}{m}\right) - \lambda_\gamma\left(\frac{2g\cos\gamma}{V^3}\right) \tag{5.54}$$

$$C_4 = -\lambda_x\sin\gamma + \lambda_h\cos\gamma - \lambda_\gamma\frac{g\sin\gamma}{V^2} \tag{5.55}$$

$$C_5 = -\lambda_x V\cos\gamma - \lambda_h V\sin\gamma + \lambda_V g\sin\gamma + \lambda_\gamma\frac{g\cos\gamma}{V} \tag{5.56}$$

$$\tilde{D} = f_p = \begin{bmatrix} V\cos\gamma & 0_{1\times 9} \\ V\sin\gamma & 0_{1\times 9} \\ \left(\frac{T}{m} - \frac{\rho V^2 S C_D}{2m} - g\sin\gamma\right) & -\frac{t_f \rho V^2 S K C_L}{m}\frac{\partial C_L}{\partial C_{L_i}} \\ \left(\frac{\rho V S C_L}{2m} - \frac{g\cos\gamma}{V}\right) & \frac{t_f \rho V S}{2m}\frac{\partial C_L}{\partial C_{L_i}} \end{bmatrix} \tag{5.57}$$

$$\tilde{E} = H_{xp} = \begin{bmatrix} 0 & 0_{1\times 9} \\ E_1 & E_2 \\ E_3 & E_4 \\ E_5 & 0_{1\times 9} \end{bmatrix} \tag{5.58}$$

where

$$E_1 = \frac{\lambda_V \rho V^2 S C_D}{2mh_s} - \frac{\lambda_\gamma \rho V S C_L}{2mh_s} \tag{5.59}$$

$$E_2 = t_f\left(\lambda_V \frac{\rho V^2 S K C_L}{mh_s} - \lambda_\gamma \frac{\rho V S}{2mh_s}\right)\frac{\partial C_L}{\partial C_{L_i}} \tag{5.60}$$

$$\begin{aligned} E_3 &= \lambda_x \cos\gamma + \lambda_h \sin\gamma - \frac{\lambda_V \rho V S C_D}{m} \\ &\quad + \frac{\lambda_\gamma \rho S C_L}{2m} - \frac{\lambda_\gamma g\cos\gamma}{V^2} \end{aligned} \tag{5.61}$$

$$E_4 = t_f\left(-\lambda_V \frac{2\rho V S K C_L}{m} + \lambda_\gamma \frac{\rho S}{2m}\right)\frac{\partial C_L}{\partial C_{L_i}} \tag{5.62}$$

$$E_5 = -\lambda_x V\sin\gamma + \lambda_h V\cos\gamma - \lambda_V g\cos\gamma + \frac{\lambda_\gamma g\sin\gamma}{V} \tag{5.63}$$

$$\tilde{F} = H_{pp} = \begin{bmatrix} 0 & F_1 \\ F_1^T & F_2 \end{bmatrix} \tag{5.64}$$

where

$$F_1 = -\frac{\lambda_V \rho V^2 S K C_L}{m}\frac{\partial C_L}{\partial C_{L_i}} + \frac{\lambda_\gamma \rho V S}{2m}\frac{\partial C_L}{\partial C_{L_i}} \tag{5.65}$$

$$F_2 = -\lambda_V \frac{\rho V^2 S K}{m}\frac{\partial C_L}{\partial C_{L_i}}\frac{\partial C_L}{\partial C_{L_j}}. \tag{5.66}$$

The boundary conditions for the Riccati equations are

$$S_f = G_{x_f x_f} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{g} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.67}$$

$$R_f = \psi_{x_f}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.68}$$

$$m_f = G_{x_f p} = \begin{bmatrix} 0 & 0_{1\times9} \\ 0 & 0_{1\times9} \\ 0 & 0_{1\times9} \\ 0 & 0_{1\times9} \end{bmatrix} \tag{5.69}$$

$$n_f = \psi_p = \begin{bmatrix} -\nu_x V_T \cos\gamma & 0_{1\times9} \\ -\nu_h V_T \sin\gamma & 0_{1\times9} \end{bmatrix} \tag{5.70}$$

$$\alpha_f = G_{pp} = \begin{bmatrix} 0 & 0_{1\times9} \\ 0_{9\times1} & 0_{9\times9} \end{bmatrix} \tag{5.71}$$

The Riccati equations (Eqs. (3.54)-(3.59)) are integrated using these quantities and the matrices V^{-1}, UV^{-1}, and $\bar{S}$ are constructed. At each integration step, these matrices are checked to make sure they are finite. In addition, at the end of the integration, the eigenvalues of the equation

$$\int_0^1 H_{C_{L_i} C_{L_j}} d\tau \tag{5.72}$$

are tested to insure positive definiteness.

Both of the trajectories described above had matrices which are finite and the eigenvalues of Eq. (5.72) are positive. The eigenvalues are listed in Table 5.1.

5.6 Summary

In this chapter, the sufficient conditions are applied to a missile intercept problem. The first variation conditions are used to get the optimal trajectory. Two different cases are investigated. The first is for the target

Case 1	*Case 2*
2.224×10^{-5}	2.052×10^{-5}
1.335×10^{-4}	1.223×10^{-4}
3.463×10^{-4}	2.527×10^{-4}
3.851×10^{-4}	3.292×10^{-4}
5.390×10^{-4}	4.066×10^{-4}
6.931×10^{-4}	5.793×10^{-4}
8.018×10^{-4}	6.923×10^{-4}
1.261×10^{-3}	1.292×10^{-3}
1.430×10^{-3}	1.596×10^{-3}

Table 5.1: Eigenvalues Of Eq.(5.72)

moving toward the launching aircraft and the second is for the target moving away from the launching aircraft. The sufficient conditions for a minimum are applied to both these trajectories and are found to be satisfied. Therefore these trajectories are minimal.

Chapter 6

First Variation Necessary Conditions for Problems with Parameterized and Nonparameterized Controls

6.1 Introduction

In this chapter the first variation necessary conditions are derived for optimal control problems with parameterized and nonparameterized controls. First, the first variation necessary conditions are derived. The first integral is obtained directly from these first variation necessary conditions.

6.2 The First Order Necessary Conditions

For the class of problems in which only a subset of the controls are to be parameterized, the optimization problem is stated as follows

$$\text{minimize } J = \phi(x_f, p) + \int_0^1 L(\tau, x, u, p)\, d\tau \tag{6.1}$$

subject to the differential constraints

$$x' = f(\tau, x, u, p) \tag{6.2}$$

and the terminal constraints

$$\psi(x_f, p) = 0 \tag{6.3}$$

with the initial conditions

$$x_0 = x_{0_s} = given \tag{6.4}$$

where, as before, p, containing the parameterized control and the final time (if it is free) is defined as

$$p \triangleq \begin{bmatrix} u_1 \\ \vdots \\ u_{k-1} \\ t_f \end{bmatrix}, \tag{6.5}$$

where, as before, $k-1$ is the number of parameterized control nodes. This formulation of the performance index will be used henceforth.

The constraints are adjoined to the performance index using Lagrange multipliers and it is expressed as

$$\bar{J} = G(x_f, p, \nu) + \int_0^1 \left[H(\tau, x, u, p, \lambda) - \lambda^T x'\right] \, d\tau \tag{6.6}$$

where

$$\begin{aligned} G(x_f, p, \nu) &\triangleq \phi(x_f, p) + \nu^T \psi(x_f, p) && (6.7) \\ H(\tau, x, u, p, \lambda) &\triangleq L(\tau, x, u, p) + \lambda^T f(\tau, x, u, p). && (6.8) \end{aligned}$$

In order to obtain the first variation necessary conditions, the variation of Eq. (6.6) is taken, yielding

$$\begin{aligned} \delta\bar{J} &= G_{x_f}\delta x_f + G_p\delta p + G_\nu \delta\nu + \int_0^1 [H_x \delta x \\ &\quad + H_u\delta u + H_\lambda \delta\lambda + H_p\delta p - \delta\lambda^T x' - \lambda^T \delta x'\big] \, d\tau. \end{aligned} \tag{6.9}$$

After the $\lambda^T \delta x'$ term is integrated by parts, the first variation becomes

$$\begin{aligned} \delta\bar{J} &= (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + G_\nu\delta\nu + +\lambda_0^T \delta x_0 \\ &\quad \int_0^1 \left[(H_x + \lambda^{T\prime})\delta x + H_u \delta u + H_p \delta p + (f^T - x^{T\prime})\delta\lambda\right] \, d\tau \end{aligned} \tag{6.10}$$

which is further simplified using Eqs. (6.2) and (6.3) (with $\delta x_0 = 0$) to

$$\delta\bar{J} = (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + \int_0^1 \left[(H_x + \lambda^{T\prime})\delta x + H_u\delta u + H_p\delta p\right] \, d\tau. \tag{6.11}$$

Now, the Lagrange multipliers, λ, are chosen such that the coefficients of the dependent variations (δx) vanish, that is

$$\lambda' = -H_x^T \tag{6.12}$$

$$\lambda_f = G_{x_f}^T. \tag{6.13}$$

The first variation now reduces to

$$\delta \bar{J} = G_p \delta p + \int_0^1 H_u \delta u + H_p \delta p \, d\tau. \tag{6.14}$$

Since the variations δp are constant, they can be removed from the integral to yield

$$\delta \bar{J} = (G_p + \int_0^1 H_p \, d\tau)\delta p + \int_0^1 H_u \delta u \, d\tau. \tag{6.15}$$

The variations δu and δp are independent. If $\delta u = 0$, the first variation must vanish because δp can be either positive or negative. Therefore, the coefficient of δp must vanish, that is,

$$G_p + \int_0^1 H_p \, d\tau = 0. \tag{6.16}$$

A similar argument can be made for δu with the additional assumption of controllability [4]. Therefore, the condition on the control u can be expressed as

$$H_u = 0. \tag{6.17}$$

As before, the quantity μ is introduced, such that Eq. (6.16) can be rewritten as follows

$$\mu' = -H_p^T(\tau, x, u, p, \lambda) \tag{6.18}$$

with the boundary conditions

$$\mu_0 = 0 \tag{6.19}$$

$$\mu_f = G_p^T. \tag{6.20}$$

To summarize, the first variation necessary conditions for a minimum are

$$x' = f(\tau, x, p) \tag{6.21}$$

$$\lambda' = -H_x^T(\tau, x, u, p, \lambda) \tag{6.22}$$

$$\mu' = -H_p^T(\tau, x, u, p, \lambda) \tag{6.23}$$

$$H_u = 0 \tag{6.24}$$

with the boundary conditions

$$\tau_0 = 0 \tag{6.25}$$

$$x_0 = x_{0_s} \tag{6.26}$$

$$\mu_o = 0 \tag{6.27}$$

$$\tau_f = 1 \tag{6.28}$$

$$\psi(x_f, p) = 0 \tag{6.29}$$

$$\lambda_f = G_{x_f}^T(x_f, p, \nu) \tag{6.30}$$

$$\mu_f = G_p^T(x_f, p, \nu). \tag{6.31}$$

For a problem with i states, j controls, k parameters, and l terminal constraints, Eqs. (6.21)-(6.24) are $2i + j + k$ equations. From Eqs (6.29)-(6.31), $l + i + k$ final conditions are obtained which allow for the solution of the constant Lagrange multipliers ν (l unknowns), the parameters p, and the final state x_f. These equations can be used to solve for the remaining constants of integration. Since the control does not enter into the endpoint function G, the condition in Eq. (6.31) can be expressed as

$$\mu_{u_{i_f}} = 0 \tag{6.32}$$

$$\mu_{t_{f_f}} = G_{t_f}. \tag{6.33}$$

6.3 The First Integral

The first integral can be calculated quite readily by noting that

$$\frac{dH}{d\tau} = H_\tau + H_x x' + H_u u' + H_\lambda \lambda' + H_p p'. \tag{6.34}$$

Since

$$x' = f,\ H_\lambda = f^T,\ \lambda' = -H_x^T,\ p' = 0,\ H_u = 0, \tag{6.35}$$

the derivative becomes

$$\frac{dH}{d\tau} = H_\tau. \tag{6.36}$$

As a result, if H does not contain τ explicitly (i.e. $H_\tau = 0$), the first integral reduces to

$$H = \text{Constant}. \tag{6.37}$$

Chapter 7

Neighboring Extremal Paths for Problems with Parameterized and Nonparameterized Control

7.1 Introduction

In Chapter 3, the neighboring extremals have been developed for problems in which the controls are parameterized. In this chapter, neighboring extremal paths are developed for problems which contain both parameterized and nonparameterized control. The analysis follows similar lines and the results are quite analogous.

7.2 Neighboring Extremals for Problems with Parameterized and Nonparameterized Control

As before, an extremal path which satisfies the first variation necessary conditions is assumed. Throughout this development, the strengthened Legendre-Clebsch condition ($H_{uu} > 0$) is assumed to be satisfied; hence, the inverse, H_{uu}^{-1}, exists.

Suppose there exists a neighboring extremal path from a point $x_{q_*} = x_q + \delta x_q$ at time t_q to a neighboring terminal constraint manifold $\psi_* = \psi + \delta\psi$. Then, an admissible comparison path is formed by letting δx_q and $\delta\psi$ go to zero. If the subscript 1 denotes a neighboring extremal path, for this to be an admissible comparison path as well, the following conditions must hold:

$$x_1{}' = f(\tau, x_1, u_1, p_1) \tag{7.1}$$

$$\lambda_1{}' = -H_x^T(\tau, x_1, u_1, \lambda_1, p_1) \tag{7.2}$$

$$0 = H_u^T(\tau, x_1, u_1, \lambda_1, p_1) \tag{7.3}$$

$$\mu_1{}' = -H_p^T(\tau, x_1, u_1, \lambda_1, p_1) \tag{7.4}$$

$$\psi(x_{1_f}, p_1) = 0,\ \lambda_{1_f} = G_{x_f}^T(x_{1_f}, p_1, \nu_1),\ \mu_{1_f} = G_p^T(x_{1_f}, p_1, \nu_1) \tag{7.5}$$

$$x_{1_0} = given,\ \mu_{1_0} = 0,\ \tau_0 = 0,\ \tau_f = 1. \tag{7.6}$$

Since $x_1(t)$ is a neighboring path to $x(t)$,

$$x_1 = x + \delta x,\quad u_1 = u + \delta u,\quad \lambda_1 = \lambda + \delta\lambda,\quad \nu_1 = \nu + \delta\nu,\quad p_1 = p + \delta p. \tag{7.7}$$

Substituting these equations into Eqs. (7.1)-(7.5), expanding in a Taylor series, and neglecting higher order terms (higher than one), the resulting equations become

$$\delta x' = f_x\delta x + f_u\delta u + f_p\delta p \tag{7.8}$$

$$\delta\lambda' = -H_{xx}\delta x - H_{xu}\delta u - f_x^T\delta\lambda - H_{xp}\delta p \tag{7.9}$$

$$0 = H_{ux}\delta x + H_{uu}\delta u + f_u^T\delta\lambda + H_{up}\delta p \tag{7.10}$$

$$\delta\mu' = -H_{px}\delta x - H_{pu}\delta u - f_p^T\delta\lambda - H_{pp}\delta p \tag{7.11}$$

$$\delta\psi = \psi_{x_f}\delta x_f + \psi_p\delta p \tag{7.12}$$

$$\delta\lambda_f = G_{x_f x_f}\delta x_f + \psi_{x_f}^T\delta\nu + G_{x_f p}\delta p \tag{7.13}$$

$$\delta\mu_f = G_{px_f}\delta x_f + \psi_p^T\delta\nu + G_{pp}\delta p. \tag{7.14}$$

where, as before, $\delta x_0 = given$ and $\delta\mu_0 = given$. In expressing Eqs. (7.9)-(7.13), the following relations have been used:

$$H_{x\lambda} = f_x^T,\quad H_{u\lambda} = f_u^T,\quad G_{x_f\nu} = \psi_{x_f}^T,\quad G_{p\nu} = \psi_p^T. \tag{7.15}$$

Eq. (7.10) can be solved for δu to yield

$$\delta u = -H_{uu}^{-1}(H_{ux}\delta x + f_u^T\delta\lambda + H_{up}\delta p). \tag{7.16}$$

Substituting this equation into Eqs. (7.8)-(7.9) and (7.10), results in

$$\delta x' = A\delta x - B\delta\lambda + D\delta p \tag{7.17}$$

$$\delta\lambda' = -C\delta x - A^T\delta\lambda - E\delta p \tag{7.18}$$

$$\delta\mu' = -E^T\delta x - D^T\delta\lambda - F\delta p \tag{7.19}$$

where

$$A = f_x - f_u H_{uu}^{-1} H_{ux} \tag{7.20}$$

$$B = f_u H_{uu}^{-1} f_u^T \tag{7.21}$$

$$C = H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} \tag{7.22}$$

$$D = f_p - f_u H_{uu}^{-1} H_{up} \tag{7.23}$$

$$E = H_{xp} - H_{xu} H_{uu}^{-1} H_{up} \tag{7.24}$$

$$F = H_{pp} - H_{pu} H_{uu}^{-1} H_{up}. \tag{7.25}$$

Eqs.(7.12)-(7.14) can be expressed as

$$\delta\lambda_f = G_{x_f x_f}\delta x_f + \psi_{x_f}^T\delta\nu + G_{x_f p}\delta p \tag{7.26}$$

$$\delta\psi = \psi_{x_f}\delta x_f + 0\delta\nu + \psi_p\delta p \tag{7.27}$$

$$\delta\mu_f = G_{p x_f}\delta x_f + \psi_p^T\delta\nu + G_{pp}\delta p. \tag{7.28}$$

7.3 Sweep Method

As in the previous section, the sweep method is used to solve this linear two-point boundary-value problem, hypothesizing that the solution is of the form of the final conditions, i.e.

$$\delta\lambda = S(\tau)\delta x + R(\tau)\delta\nu + m(\tau)\delta p \tag{7.29}$$

$$\delta\psi = R^T(\tau)\delta x + Q(\tau)\delta\nu + n(\tau)\delta p \tag{7.30}$$

$$\delta\mu = m^T(\tau)\delta x + n^T(\tau)\delta\nu + \alpha(\tau)\delta p \tag{7.31}$$

where

$$
\begin{aligned}
S_f &= G_{x_f x_f} && (7.32)\\
R_f &= \psi_{x_f}^T && (7.33)\\
m_f &= G_{x_f p} && (7.34)\\
Q_f &= 0 && (7.35)\\
n_f &= \psi_p && (7.36)\\
\alpha_f &= G_{pp}. && (7.37)
\end{aligned}
$$

Differentiating Eq. (7.29) yields

$$\delta\lambda' = S'\delta x + S\delta x' + R'\delta\nu + m'\delta p. \tag{7.38}$$

Substituting for $\delta x'$ and $\delta\lambda'$ from Eqs. (7.17) and (7.18), respectively, yields,

$$-C\delta x - A^T\delta\lambda - E\delta p = S'\delta x + S(A\delta x - B\delta\lambda + D\delta p) + R'\delta\nu + m'\delta p. \tag{7.39}$$

Substituting for $\delta\lambda$ from Eq. (7.29), and collecting terms produces

$$
\begin{aligned}
0 &= (S' + C + SA + A^T S - SBS)\delta x + (R' + A^T R - SBR)\delta\nu\\
&\quad +(m' + E + SD + A^T m - SBm)\delta p. && (7.40)
\end{aligned}
$$

Since this equation is valid for all variations in x, ν, and p, each of the coefficients of the variations must be zero, so S, R, and m are chosen such that

$$
\begin{aligned}
S' &= SBS - A^T S - SA - C, && S_f = G_{x_f x_f} && (7.41)\\
R' &= (SB - A^T)R, && R_f = \psi_{x_f}^T && (7.42)\\
m' &= (SB - A^T)m - (SD + E), && m_f = G_{x_f p}. && (7.43)
\end{aligned}
$$

Notice that since B, C, and $G_{x_f x_f}$ are all symmetric, and since the differential equation for S^T and the boundary condition are the same as for S, $S = S^T$, which means that S is symmetric.

Similarly, differentiating Eq. (7.30), with $\delta\psi' = 0$, gives

$$0 = R^{T\prime}\delta x + R^T\delta x' + Q'\delta\nu + n'\delta p. \tag{7.44}$$

Substituting for $\delta x'$, yields

$$0 = R^{T\prime}\delta x + R^T(A\delta x - B\delta\lambda + D\delta p) + Q'\delta\nu + n'\delta p. \tag{7.45}$$

Accumulating the coefficients of the common variations produces

$$0 = (R^{T\prime} + R^T A - R^T BS)\delta x + (Q' - R^T BR)\delta\nu + (n' - R^T Bm + TD)\delta p. \tag{7.46}$$

If R, Q, and m are chosen such that the coefficients of the variations vanish, this results in the following set of equations with the associated boundary conditions

$$R' = (SB - A^T)R, \qquad R_f = \psi_{x_f}^T \tag{7.47}$$

$$Q' = R^T BR, \qquad Q_f = 0 \tag{7.48}$$

$$n' = R^T Bm - R^T D, \qquad n_f = \psi_p. \tag{7.49}$$

Finally, differentiating Eq. (7.31), yields,

$$\delta\mu' = m^{T\prime}\delta x + m^T\delta x' + n^{T\prime}\delta\nu + \alpha'\delta p. \tag{7.50}$$

Substituting for $\delta x'$ and $\delta\mu'$ from Eqs. (7.17) and (7.19), respectively, yields

$$-E^T\delta x - D^T\delta\lambda - F\delta p = m^{T\prime}\delta x + m^T(A\delta x - B\delta\lambda + D\delta p) + n^T\delta\nu + \alpha'\delta p. \tag{7.51}$$

Substituting for $\delta\lambda$ from Eq. (7.29), and collecting terms produces

$$\begin{aligned} 0 &= (m^{T\prime} + E^T + m^T A - m^T BS + D^T S)\delta x + (n^{T\prime} + D^T R - m^T BR)\delta\nu \\ &\quad +(\alpha' + F + m^T D + D^T m - m^T Bm)\delta p. \end{aligned} \tag{7.52}$$

This yields the three equations

$$
\begin{aligned}
m' &= (SB - A^T)m - (SD + E), \quad & m_f = G_{x_f p} \qquad (7.53)\\
n' &= R^T Bm - R^T D, \quad & n_f = \psi_p \qquad (7.54)\\
\alpha' &= m^T Bm - D^T m - m^T D - F, \quad & \alpha_f = G_{pp}. \qquad (7.55)
\end{aligned}
$$

To summarize, the differential equations for S, R, Q, m, n, and α and their respective boundary conditions are

$$
\begin{aligned}
S' &= SBS - A^T S - SA - C, \quad & S_f = G_{x_f x_f} \qquad (7.56)\\
R' &= (SB - A^T)R, \quad & R_f = \psi^T_{x_f} \qquad (7.57)\\
Q' &= R^T BR, \quad & Q_f = 0 \qquad (7.58)\\
m' &= (SB - A^T)m - (SD + E), \quad & m_f = G_{x_f p} \qquad (7.59)\\
n' &= R^T Bm - R^T D, \quad & n_f = \psi_p \qquad (7.60)\\
\alpha' &= m^T Bm - D^T m - m^T D - F, \quad & \alpha_f = G_{pp}. \qquad (7.61)
\end{aligned}
$$

7.4 The Neighboring Extremal Control

Eqs. (7.29)-(7.31) can be written in matrix form as

$$
\begin{bmatrix} \delta\lambda \\ \delta\psi \\ \delta\mu \end{bmatrix} = \begin{bmatrix} S(\tau) & R(\tau) & m(\tau) \\ R^T(\tau) & Q(\tau) & n(\tau) \\ m^T(\tau) & n^T(\tau) & \alpha(\tau) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\nu \\ \delta p \end{bmatrix}. \qquad (7.62)
$$

The last two equations of the matrix can be written in the form

$$
\begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix} = U^T \delta x + V \begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix} \qquad (7.63)
$$

where

$$
\begin{aligned}
U &\triangleq \begin{bmatrix} R(\tau) & m(\tau) \end{bmatrix}, \qquad (7.64)\\
V &\triangleq \begin{bmatrix} Q(\tau) & n(\tau) \\ n^T(\tau) & \alpha(\tau) \end{bmatrix}. \qquad (7.65)
\end{aligned}
$$

Therefore, if V^{-1} exists,

$$\begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix} = -V^{-1}U^T\delta x + V^{-1}\begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{7.66}$$

Eq.(7.29) can be rewritten as

$$\delta\lambda = S\delta x + U\begin{bmatrix} \delta\nu \\ \delta p \end{bmatrix}. \tag{7.67}$$

Use of Eq. (7.66) results in the following equation:

$$\delta\lambda = (S - UV^{-1}U^T)\delta x + UV^{-1}\begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{7.68}$$

As before, taking advantage of the symmetry of V, V^{-1} is defined as

$$V^{-1} \triangleq \begin{bmatrix} a & b \\ b^T & c \end{bmatrix} \tag{7.69}$$

where a is of the same dimension as Q $(l \times l)$, b is of the same dimension as n $(l \times k)$, and c is the same dimension as α $(k \times k)$. In addition, $\bar{S}$ is defined as

$$\bar{S} \triangleq S - UV^{-1}U^T \tag{7.70}$$

and $\delta\Phi$ is defined as

$$\delta\Phi \triangleq \begin{bmatrix} \delta\psi \\ \delta\mu \end{bmatrix}. \tag{7.71}$$

With the definitions of V^{-1} and U, the variations $\delta\nu$ and δp can be written as

$$\delta\nu = -(aR^T + bm^T)\delta x + b\delta\mu + a\delta\psi \tag{7.72}$$

$$\delta p = -(b^TR^T + cm^T)\delta x + c\delta\mu + b^T\delta\psi. \tag{7.73}$$

As in the previous section, the quantities $Rb + mc$ and $Ra + mb^T$ appear consistently and are defined as

$$M \triangleq b^TR^T + cm^T = \begin{bmatrix} b^T & c \end{bmatrix} U^T \tag{7.74}$$

$$N \triangleq aR^T + bm^T = \begin{bmatrix} a & b \end{bmatrix} U^T. \tag{7.75}$$

Therefore, $\delta\nu$ and δp can be expressed as

$$\delta p = -M\delta x + c\delta\mu + b^T\delta\psi \tag{7.76}$$

$$\delta\nu = -N\delta x + b\delta\mu + a\delta\psi. \tag{7.77}$$

This result, along with Eqs. (7.76) and (7.68), is used to express δu from Eq. (7.16) as

$$\begin{aligned}\delta u = & -H_{uu}^{-1}\left\{\left[H_{ux} + f_u^T\bar{S} - H_{up}M\right]\delta x + \left[f_u^T M^T + H_{up}c\right]\delta\mu\right. \\ & \left. + \left[f_u^T N^T + H_{up}b^T\right]\delta\psi\right\}.\end{aligned} \tag{7.78}$$

In addition, if the definitions of $\bar{S}$, V^{-1}, M, and N are used, the differential equation for $\delta\mu$ can be expressed as follows

$$\begin{aligned}\delta\mu' = & -\left(E^T + D^T\bar{S} - FM\right)\delta x - \left(D^T N^T + Fb^T\right)\delta\psi \\ & -\left(D^T M^T + Fc\right)\delta\mu\end{aligned} \tag{7.79}$$

The differential equations for $\bar{S}$, N, M, a, b, and c are derived in Appendix B.

Chapter 8

Sufficient Conditions for Problems with Parameterized and Nonparameterized Control

8.1 Introduction

The sufficient conditions for minima can be used to determine whether a particular set of parameterized and nonparameterized controls, which satisfy the first variation necessary conditons, will minimize the performance index. As before, the second variation is derived from the first variation. Then, the sufficient conditions are expressed in terms of the Jacobi (conjugate point) conditions.

8.2 The Second Variation

Recall that the first variation was expressed in Eq. (2.21) as

$$\begin{aligned} \delta\bar{J} &= (G_{x_f} - \lambda_f^T)\delta x_f + G_p\delta p + G_\nu\delta\nu + \lambda_0^T\delta x_0 \\ &+ \int_0^1 (H_x - \lambda')\delta x + H_u\delta u + H_p\delta p + (f^T - x^{T'})\delta\lambda \, d\tau. \end{aligned} \tag{8.1}$$

The second variation is obtained by taking the variation of the first variation, that is

$$\begin{aligned} \delta^2\bar{J} &= \delta x_f^T G_{x_f x_f}\delta x_f + \delta x_f^T G_{x_f p}\delta p - \delta x_f^T\delta\lambda_f + \delta p^T G_{p x_f}\delta x_f \\ &+ \lambda_0^T\delta^2 x_0 + \delta p^T G_{pp}\delta p + \delta\psi^T\delta\nu + (\delta x^T G_{x_f\nu} + \delta p^T G_{p\nu})\delta\nu \\ &+ \delta\lambda_0^T\delta x_0 + \lambda_0^T\delta^2 x_0 + \delta\nu^T\left(G_{\nu x_f}\delta x_f + G_{\nu p}\delta p\right) \\ &+ \int_0^1 (H_x + \lambda^{T'})\delta^2 x + \delta x^T H_{xx}\delta x + \delta x^T H_{xu}\delta u \end{aligned} \tag{8.2}$$

$$+\delta x^T H_{x\lambda}\delta\lambda + \delta x^T H_{xp}\delta p + \delta x^T \delta\lambda' + \delta u^T H_{ux}\delta x$$
$$+\delta u^T H_{uu}\delta u + \delta u^T H_{u\lambda}\delta\lambda + \delta u^T H_{up}\delta p + \delta p^T H_{px}\delta x$$
$$+\delta p^T H_{pu}\delta u + \delta p^T H_{p\lambda}\delta\lambda + \delta p^T H_{pp}\delta p + (f^T - x^{T'})\delta^2\lambda$$
$$+\delta\lambda^T(f_x\delta x + f_u\delta u + f_p\delta p - \delta x')\, d\tau. \tag{8.3}$$

If the term $\delta x^T\delta\lambda'$ is integrated by parts and Eqs. (6.21), (6.22), and (7.15), as well as the constraints are used (with $\delta^2 x_0 = 0$), the second variation is reduced to

$$\delta^2\bar{J} = \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix}$$
$$+ \int_0^1 \begin{bmatrix} \delta x^T & \delta u^T & \delta p^T \end{bmatrix} \begin{bmatrix} H_{xx} & H_{xu} & H_{xp} \\ H_{ux} & H_{uu} & H_{up} \\ H_{px} & H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \\ \delta p \end{bmatrix} d\tau. \tag{8.4}$$

8.3 The Perfect Differential

Now, a perfect differential of the form,

$$P = \frac{d}{d\tau}\left[\delta x^T \bar{S}\delta x + \delta x^T U V^{-1}\delta\Phi + \delta\Phi^T V^{-1} U^T \delta x - \delta\Phi^T V^{-1}\delta\Phi\right] \tag{8.5}$$
$$= \frac{d}{d\tau}\left[\delta x^T \bar{S}\delta x + \delta x^T M^T \delta\mu + \delta\mu^T M\delta x + \delta x^T N^T \delta\psi\right.$$
$$\left. + \delta\psi^T N\delta x - \delta\psi^T a\,\delta\psi - \delta\psi^T b\,\delta\mu - \delta\mu^T b^T\delta\psi - \delta\mu^T c\,\delta\mu\right] \tag{8.6}$$

along with the condition

$$\delta x' = f_x\delta x + f_u\delta u + f_p\delta p \tag{8.7}$$

is expanded in the following manner. If Eq. (8.7) and the definitions of A-F from Eqs. (7.20)-(7.25) are substituted into Eq. (8.6) and factored, P becomes

$$P = \delta x^T\left(A^T\bar{S} + \bar{S}A + H_{xu}H_{uu}^{-1}f_u^T\bar{S} + \bar{S}f_u H_{uu}^{-1}H_{ux} + \bar{S}'\right)\delta x$$
$$+\delta u^T f_u^T \bar{S}\delta x + \delta x^T \bar{S} f_u \delta u + \delta p^T f_p^T \bar{S}\delta x$$

$$
\begin{aligned}
&+\delta x^T \bar{S} f_p \delta p + \delta x^T \Big(M^{T\prime} + A^T M^T \\
&+ H_{xu} H_{uu}^{-1} f_u^T M^T \Big) \delta\mu + \delta u^T f_u^T M^T \delta\mu + \delta\mu^T M f_u \delta u \\
&+\delta\mu^T \Big(M' + MA + M f_u H_{uu}^{-1} H_{ux} \Big) \delta x \\
&+\delta p^T f_p^T M^T \delta\mu + \delta\mu^T M f_p \delta p + \delta p^T f_p^T N^T \delta\psi \\
&+\delta\psi^T N f_p \delta p + \delta\psi^T \Big(N' + NA + N f_u H_{uu}^{-1} H_{ux} \Big) \delta x \\
&+\delta u^T f_u^T N^T \delta\psi + \delta\psi^T N f_u \delta u \\
&+\delta x^T \Big(N^{T\prime} + A^T N^T + H_{xu} H_{uu}^{-1} f_u^T N^T \Big) \delta\psi \\
&-\delta\mu^{T\prime} \Big(-M\delta x + b^T \delta\psi + c\delta\mu \Big) - \delta\psi^T a' \delta\psi - \delta\psi^T b' \delta\mu \\
&- \Big(-\delta x^T M^T + \delta\psi^T b + \delta\mu^T c \Big) \delta\mu' - \delta\mu^T b^{T\prime} \delta\psi - \delta\mu^T c' \delta\mu. \qquad (8.8)
\end{aligned}
$$

The quantity P is further expanded, by adding and subtracting certain terms, it becomes

$$
\begin{aligned}
P \;=\; & \delta x^T \Big[A^T \bar{S} + \bar{S} A + H_{xu} H_{uu}^{-1} f_u^T \bar{S} + \bar{S} f_u H_{uu}^{-1} H_{ux} + \bar{S}' + C \\
& - H_{xx} + H_{xu} H_{uu}^{-1} H_{ux} - \bar{S} B \bar{S} - EM - M^T E^T + M^T F M \\
& - \bar{S} D M - M^T D^T \bar{S} - M^T H_{pu} H_{uu}^{-1} f_u^T \bar{S} + \bar{S} f_u H_{uu}^{-1} f_u^T \bar{S} \\
& - \bar{S} f_u H_{uu}^{-1} H_{up} M - H_{xu} H_{uu}^{-1} H_{up} M + M H_{pu} H_{uu}^{-1} H_{up} M \\
& - M^T H_{pu} H_{uu}^{-1} H_{ux} + M^T H_{pp} M \Big] \delta x + \delta u^T \Big(f_u^T \bar{S} - H_{up} M \Big) \delta x \\
& + \delta x^T \Big(\bar{S} f_u - M^T H_{pu} \Big) \delta u + \delta u^T H_{up} \Big(M \delta x - c \delta\mu - b^T \delta\psi \Big) \\
& + \Big(\delta x^T M^T - \delta\mu^T c - \delta\psi^T b \Big) H_{pu} \delta u + \delta x^T \bar{S} f_p \delta p + \delta p^T f_p^T \bar{S} \delta x \\
& + \delta\mu^T M f_p \delta p + \delta p^T f_p^T M^T \delta\mu + \delta x^T \Big[N^{T\prime} - \bar{S} B N^T + A^T N^T \\
& + \bar{S} f_u H_{uu}^{-1} f_u^T N^T + \bar{S} D B^T - M^T D^T N^T - M^T F b^T + E b^T \\
& + \bar{S} f_u H_{uu}^{-1} H_{up} b^T + H_{xu} H_{uu}^{-1} f_u^T N^T + H_{xu} H_{uu}^{-1} H_{up} b^T \\
& - M^T H_{pu} H_{uu}^{-1} f_u^T N^T - M^T H_{pu} H_{uu}^{-1} H_{up} b^T - M H_{pp} b^T \Big] \delta\psi \\
& + \delta\psi^T \Big[N' - N B \bar{S} + f_u H_{uu}^{-1} f_u^T \bar{S} + NA - N D^T \bar{S} + b E^T
\end{aligned}
$$

$$+NDM - bFM + Nf_u H_{uu}^{-1} H_{ux} + bH_{pu} H_{uu}^{-1} H_{ux}$$

$$+bH_{pu} H_{uu}^{-1} f_u^T \bar{S} - Nf_u H_{uu}^{-1} H_{up} M - bH_{pu} H_{uu}^{-1} H_{up} M$$

$$-bH_{pp} M^T \Big] \delta x + \delta x^T \Big[M^{T\prime} + A^T M^T + H_{xu} H_{uu}^{-1} f_u^T M^T + Ec$$

$$-\bar{S} B M^T + \bar{S} Dc - M^T D^T M^T - M^T Fc + \bar{S} f_u H_{uu}^{-1} f_u^T M^T$$

$$+H_{xu} H_{uu}^{-1} H_{up} c + \bar{S} f_u H_{uu}^{-1} H_{up} c - M^T H_{pu} H_{uu}^{-1} H_{up} c$$

$$+M^T H_{pu} H_{uu}^{-1} f_u^T M^T - MH_{pp} c \Big] \delta\mu + \delta\mu^T \left[M' + MA \right.$$

$$+Mf_u H_{uu}^{-1} H_{ux} - MB\bar{S} + cD^T \bar{S} - MDM - cFM$$

$$+cE^T + Mf_u H_{uu}^{-1} f_u^T \bar{S} + cH_{pu} H_{uu}^{-1} H_{ux} + cH_{pu} H_{uu}^{-1} f_u^T \bar{S}$$

$$-cH_{pu} H_{uu}^{-1} H_{up} M - Mf_u H_{uu}^{-1} H_{up} M - cH_{pp} M \Big] \delta x$$

$$+\delta\psi^T (bH_{pu} + Nf_u) \delta u + \delta u^T \left(H_{up} b^T + f_u^T N^T \right) \delta\psi$$

$$+\delta\mu^T (cH_{pu} + Mf_u) \delta u + \delta u^T \left(H_{up} c + f_u^T M^T \right) \delta\mu$$

$$+\delta\psi^T \Big[-b' - NBM^T + NDc + bD^T M^T + bFc + bH_{pp} c$$

$$+bH_{pu} H_{uu}^{-1} f_u^T M^T + Nf_u H_{uu}^{-1} H_{up} c + Nf_u H_{uu}^{-1} f_u^T M^T$$

$$+bH_{pu} H_{uu}^{-1} H_{up} c \Big] \delta\mu + \delta\mu^T \Big[-b^{T\prime} - MBN^T + cD^T N^T$$

$$+MDb^T + cFb^T + cH_{pu} H_{uu}^{-1} f_u^T N^T + Mf_u H_{uu}^{-1} H_{up} b^T$$

$$+Mf_u H_{uu}^{-1} f_u^T N^T + cH_{pu} H_{uu}^{-1} H_{up} b^T + cH_{pp} b^T \Big] \delta\psi$$

$$+\delta\psi^T \Big[-a' - NBN^T + NDb^T + bD^T N^T + bFb^T$$

$$+bH_{pu} H_{uu}^{-1} f_u^T N^T + Nf_u H_{uu}^{-1} H_{up} b^T + Nf_u H_{uu}^{-1} f_u^T N^T$$

$$+bH_{pu} H_{uu}^{-1} H_{up} b^T + bH_{pp} b^T \Big] \delta\psi + \delta\mu^T \Big[-c' - MBM^T$$

$$+MDc + cD^T M^T + cFc + cH_{pp} c + cH_{pu} H_{uu}^{-1} f_u^T M^T$$

$$+Mf_u H_{uu}^{-1} H_{up} c + Mf_u H_{uu}^{-1} f_u^T M^T + cH_{pu} H_{uu}^{-1} H_{up} c \Big] \delta\mu$$

$$-\delta p^T f_p^T N^T \delta\psi - \delta\psi^T N f_p \delta p - \Big[\delta x^T \left(H_{xu} + \bar{S} f_u - M^T H_{up} \right)$$

$$+\delta\mu^T (Mf_u + cH_{pu}) + \delta\psi^T (Nf_u + bH_{pu}) \Big] H_{uu}^{-1} H_{up} \cdot$$

$$\cdot \left(-M\delta x + b^T \delta\psi + c\delta\mu \right) - \left(-\delta x^T M^T + \delta\psi^T b + \delta\mu^T c \right) \cdot$$

$$
\begin{aligned}
&\cdot H_{pu} H_{uu}^{-1} \left[\left(H_{ux} + f_u^T \bar{S} - H_{up} M \right) \delta x \right. \\
&+ \left(f_u^T M^T + H_{up} c \right) \delta \mu + \left. \left(f_u^T N^T + H_{up} b^T \right) \delta \psi \right] \\
&- \left[\delta \mu^{T\prime} + \delta x^T \left(E + \bar{S} D - M^T F \right) + \delta \psi^T \left(bF + ND \right) \right. \\
&\left. + \delta \mu^T \left(cF + MD \right) \right] \left(-M \delta x + b^T \delta \psi + c \delta \mu \right) \\
&+ \left(-\delta x^T M^T + \delta \psi^T b + \delta \mu^T c \right) \left[\delta \mu' . + \left(E^T + D^T \bar{S} - FM \right) \delta x \right. \\
&+ \left. \left(F b^T + D^T N^T \right) \delta \psi + \left(Fc + D^T M^T \right) \delta \mu \right] \qquad (8.9)
\end{aligned}
$$

If certain terms in P are recognized as the neighboring extremal control and parameters,

$$
\begin{aligned}
\delta \bar{u} &= -H_{uu}^{-1} \left[\left(H_{ux} + f_u^T \bar{S} - H_{up} M \right) \delta x \right. \\
&\quad \left. \left(f_u^T M^T + H_{up} c \right) \delta \mu + \left(f_u^T N^T + H_{up} b^T \right) \delta \psi \right] \qquad (8.10) \\
\delta \bar{p} &= -M \delta x + c \delta \mu + b^T \delta \psi, \qquad (8.11)
\end{aligned}
$$

and P and the quantity $\left(\delta \mu^{T\prime} \delta p + \delta p^T \delta \mu' \right)$ are added to and subtracted from the second variation defined in Eq.(8.4), $\delta^2 J$ becomes,

$$
\begin{aligned}
\delta^2 \bar{J} &= \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\
&+ \int_0^1 \left\{ \begin{bmatrix} \left(\delta u^T - \delta \bar{u}^T \right) & \left(\delta p^T - \delta \bar{p}^T \right) \end{bmatrix} \begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta \bar{u}) \\ (\delta p - \delta \bar{p}) \end{bmatrix} \right. \\
&+ \delta x^T \left[\bar{S}' + A^T \bar{S} + \bar{S} A + C - \bar{S} B \bar{S} - EM - M^T E^T + M^T F M \right. \\
&\left. - \bar{S} D M - M^T D^T \bar{S} \right] \delta x + \delta x^T \left[N^{T\prime} - \bar{S} B N^T \right. \\
&\left. + A^T N^T + \bar{S} D b^T - M^T D^T N^T - M^T F b^T + E b^T \right] \delta \psi \\
&+ \delta \psi \left[N' - N B \bar{S} + N A + b D^T \bar{S} - N D M - b F M + b E^T \right] \delta x \\
&+ \delta x^T \left[M^{T\prime} + A^T M^T - \bar{S} B M^T + \bar{S} D c - M^T D^T M^T - M^T F c + E c \right] \delta \mu \\
&+ \delta \mu^T \left[M' + M A - M B \bar{S} + c D^T \bar{S} - M D M - c F M \right. \\
&\left. + c E^T \right] \delta x - \delta \psi^T \left[a' + N B N^T - N D b^T - b D^T N^T - b F b^T \right] \delta \psi
\end{aligned}
$$

$$
\begin{aligned}
&-\delta\psi^T\left[b'+NBM^T-NDc-bD^TM^T-bFc\right]\delta\mu\\
&-\delta\mu^T\left[b^{T\prime}+MBN^T-MDb^T-cD^TN^T-cFb^T\right]\delta\psi\\
&-\delta\mu^T\left[c'+MBM^T-MDc-cD^TM^T-cFc\right]\delta\mu\\
&+\left[\delta\mu^{T\prime}+\delta x^T\left(E+\bar{S}D-M^TF\right)\right.\\
&\left.+\delta\psi^T\left(bF+ND\right)+\delta\mu^T\left(cF+MD\right)\right]\left(\delta p-\delta\bar{p}\right)\\
&+\left(\delta p^T-\delta\bar{p}^T\right)\left[\delta\mu'.+\left(E^T+D^T\bar{S}-FM\right)\delta x\right.\\
&\left.\left.+\left(Fb^T+D^TN^T\right)\delta\psi+\left(Fc+D^TM^T\right)\delta\mu\right]\right\}\,d\tau. \qquad (8.12)
\end{aligned}
$$

Now, $\bar{S}$, M, N, a, b, c, and $\delta\mu$ are chosen to satisfy the following relations, similar to the reasoning in Bryson and Ho[4],

$$
\begin{aligned}
\bar{S}' &= \bar{S}B\bar{S}-A^T\bar{S}-\bar{S}A-C+EM+M^TE^T\\
&\quad -M^TFM+\bar{S}DM+M^TD^T\bar{S} && (8.13)\\
M' &= MB\bar{S}-MA-cD^T\bar{S}+MD^TM^T+cFM^T-cE^T && (8.14)\\
N' &= NB\bar{S}-NA-bD^T\bar{S}+ND^TM^T+bFM^T-bE^T && (8.15)\\
a' &= -NBN^T+NDb^T+bD^TN^T+bFb^T && (8.16)\\
b' &= -NBM^T+NDc+bD^TM^T+bFc && (8.17)\\
c' &= -MBM^T+MDc+cD^TM^T+cFc && (8.18)\\
\delta\mu' &= -\left(E^T+D^T\bar{S}-FM\right)\delta x\\
&\quad -\left(Fb^T+D^TN^T\right)\delta\psi-\left(Fc+D^TM^T\right)\delta\mu && (8.19)
\end{aligned}
$$

with the following boundary conditions

$$
\begin{aligned}
\bar{S}_f &= G_{x_fx_f}-\begin{bmatrix}\psi_{x_f}^T & G_{x_fp}\end{bmatrix}\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1}\begin{bmatrix}\psi_{x_f}\\ G_{px_f}\end{bmatrix} && (8.20)\\
\begin{bmatrix}N_f^T & M_f^T\end{bmatrix} &= U_fV_f^{-1}=\begin{bmatrix}\psi_{x_f}^T & G_{x_fp}\end{bmatrix}\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1} && (8.21)\\
V_f^{-1} &= \begin{bmatrix}a_f & b_f\\ b_f^T & c_f\end{bmatrix}=\begin{bmatrix}0 & \psi_p\\ \psi_p^T & G_{pp}\end{bmatrix}^{-1} && (8.22)
\end{aligned}
$$

$$\delta\mu_f = G_{px_f}\delta x_f + \psi_p^T\delta\bar{\nu} + G_{pp}\delta\bar{p}, \tag{8.23}$$

where the bar over p and ν denotes those quantities correspond to the neighboring extremal trajectory.

The second variation now becomes

$$\begin{aligned}
\delta^2\bar{J} &= \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\
&+ \int_0^1 \Big\{ \begin{bmatrix} \left(\delta u^T - \delta\bar{u}^T\right) & \left(\delta p^T - \delta\bar{p}^T\right) \end{bmatrix} \begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta\bar{u}) \\ (\delta p - \delta\bar{p}) \end{bmatrix} \\
&- \delta p^T\delta\mu' - \delta\mu^{T\prime}\delta p - \frac{d}{d\tau}\Big[\delta x^T\bar{S}\delta x \\
&+ \delta x^T U V^{-1}\delta\Phi + \delta\Phi^T V^{-1}U^T\delta x - \delta\Phi^T V^{-1}\,\delta\Phi\Big]\Big\}\,d\tau.
\end{aligned} \tag{8.24}$$

The second, third, and fourth terms can be integrated directly; therefore, the second variation is further reduced to

$$\begin{aligned}
\delta^2\bar{J} &= \begin{bmatrix} \delta x_f^T & \delta p^T \end{bmatrix} \begin{bmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{bmatrix} \begin{bmatrix} \delta x_f \\ \delta p \end{bmatrix} \\
&- \Big[\delta p^T\delta\mu + \delta\mu^T\delta p\Big]_0^1 - \Big[\delta x^T\bar{S}\delta x + \delta x^T U V^{-1}\delta\Phi \\
&+ \delta\Phi^T V^{-1}U^T\delta x - \delta\Phi^T V^{-1}\,\delta\Phi\Big]_0^1 \\
&+ \int_0^1 \begin{bmatrix} \left(\delta u^T - \delta\bar{u}^T\right) & \left(\delta p^T - \delta\bar{p}^T\right) \end{bmatrix} \\
&\quad \begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta\bar{u}) \\ (\delta p - \delta\bar{p}) \end{bmatrix} d\tau.
\end{aligned} \tag{8.25}$$

The terms outside the integral are separated into those at the initial time and those at the final time as follows:

$$\begin{aligned}
\delta^2\bar{J} &= Y_o + Y_f + \int_0^1 \begin{bmatrix} \left(\delta u^T - \delta\bar{u}^T\right) & \left(\delta p^T - \delta\bar{p}^T\right) \end{bmatrix} \\
&\quad \begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta\bar{u}) \\ (\delta p - \delta\bar{p}) \end{bmatrix} d\tau
\end{aligned} \tag{8.26}$$

where Y_o and Y_f are defined by

$$Y_o = \delta p^T\delta\mu_o + \delta\mu_o^T\delta p + \delta x_o^T\bar{S}_o\delta x_o + \delta x_o^T U_o V_o^{-1}\delta\Phi_o$$

$$+\delta\Phi_o^T V_o^{-1} U_o^T \delta x_o - \delta\Phi_o^T V_o^{-1} \delta\Phi_o \tag{8.27}$$

$$\begin{aligned} Y_f &= \delta x_f^T G_{x_f x_f} \delta x_f + \delta x_f^T G_{x_f p} \delta p + \delta p^T G_{p x_f} \delta x_f + \delta p^T G_{pp} \delta p \\ &\quad - \delta p^T \delta\mu_f - \delta\mu_f^T \delta p - \delta x_f^T \bar{S}_f \delta x_f - \delta x_f^T U_f V_f^{-1} \delta\Phi_f \\ &\quad - \delta\Phi_f^T V_f^{-1} U_f^T \delta x_f + \delta\Phi_f^T V_f^{-1} \delta\Phi_f. \end{aligned} \tag{8.28}$$

After substituting for μ_f, $\bar{S}_f$, $U_f V_f^{-1}$, and V_f^{-1}, Eq. (8.28) is rewritten as

$$\begin{aligned} Y_f &= \delta p^T G_{pp} \delta p - \delta p^T \left[\psi_p^T \delta\bar{\nu} + G_{pp}\delta\bar{p}\right] - \left[\delta\bar{\nu}^T \psi_p + \delta\bar{p} G_{pp}\right] \delta p \\ &\quad + \delta x_f^T \left\{ \begin{bmatrix} \psi_{x_f}^T & G_{x_f p} \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \psi_{x_f} \\ G_{p x_f} \end{bmatrix} \right\} \delta x_f \\ &\quad - \delta x_f^T \begin{bmatrix} \psi_{x_f}^T & G_{x_f p} \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix} \\ &\quad - \begin{bmatrix} \delta\psi^T & \delta\mu_f^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \psi_{x_f} \\ G_{p x_f} \end{bmatrix} \delta x_f \\ &\quad + \begin{bmatrix} \delta\psi^T & \delta\mu_f^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix}. \end{aligned} \tag{8.29}$$

Recall that on the neighboring extremal path, Eq. (7.63) evaluated at the final time is expressed as

$$\begin{bmatrix} \psi_{x_f} \\ G_{p x_f} \end{bmatrix} \delta x_f = - \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta\bar{\nu} \\ \delta\bar{p} \end{bmatrix} + \begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix} \tag{8.30}$$

Eq. (8.30) is used to reduce Eq.(8.29) to

$$\begin{aligned} Y_f &= \delta p^T G_{pp} \delta p - \delta p^T G_{pp} \delta\bar{p} - \delta\bar{p}^T G_{pp} \delta p - \delta p^T \psi_p^T \delta\bar{\nu} - \delta\bar{\nu}^T \psi_p \delta p \\ &\quad + \left\{ - \begin{bmatrix} \delta\bar{\nu}^T & \delta\bar{p}^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} + \begin{bmatrix} \delta\psi^T \\ \delta\mu_f^T \end{bmatrix} \right\} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \cdot \\ &\quad \cdot \left\{ - \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} \begin{bmatrix} \delta\bar{\nu} \\ \delta\bar{p} \end{bmatrix} + \begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix} \right\} \\ &\quad - \left\{ - \begin{bmatrix} \delta\bar{\nu}^T & \delta\bar{p}^T \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix} + \begin{bmatrix} \delta\psi^T & \delta\mu_f^T \end{bmatrix} \right\} \\ &\quad \cdot \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \delta\psi^T \\ \delta\mu_f^T \end{bmatrix} - \begin{bmatrix} \delta\psi & \delta\mu_f \end{bmatrix} \begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1} \cdot \end{aligned}$$

$$\cdot\left\{-\begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}\begin{bmatrix} \delta\bar{\nu} \\ \delta\bar{p} \end{bmatrix}+\begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix}\right\}$$

$$+\begin{bmatrix} \delta\psi^T & \delta\mu_f^T \end{bmatrix}\begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}^{-1}\begin{bmatrix} \delta\psi \\ \delta\mu_f \end{bmatrix} \tag{8.31}$$

which further simplifies to

$$\begin{aligned} Y_f &= \delta p^T G_{pp}\delta p - \delta p^T G_{pp}\delta\bar{p} - \delta\bar{p}^T G_{pp}\delta p - \delta p^T \psi_p^T \delta\bar{\nu} \\ &\quad -\delta\bar{\nu}^T \psi_p \delta p + \begin{bmatrix} \delta\bar{\nu}^T & \delta\bar{p}^T \end{bmatrix}\begin{bmatrix} 0 & \psi_p \\ \psi_p^T & G_{pp} \end{bmatrix}\begin{bmatrix} \delta\bar{\nu} \\ \delta\bar{p} \end{bmatrix}. \end{aligned} \tag{8.32}$$

Y_f now reduces to

$$\begin{aligned} Y_f &= (\delta p - \delta\bar{p})^T G_{pp}(\delta p - \delta\bar{p}) - (\delta p - \delta\bar{p})^T \psi_p^T \delta\bar{\nu} \\ &\quad -\delta\bar{\nu}^T \psi_p (\delta p - \delta\bar{p}). \end{aligned} \tag{8.33}$$

At the terminal constraint manifold, the neighboring extremal and the admissible comparison path, respectively, must satisfy the following equations

$$0 = \psi_{x_f}\delta x_f + \psi_p \delta\bar{p} \tag{8.34}$$

$$0 = \psi_{x_f}\delta x_f + \psi_p \delta p, \tag{8.35}$$

Therefore, if these two equations are subtracted, the resulting equation is

$$\psi_p(\delta p - \delta\bar{p}) = 0 \tag{8.36}$$

With this relation, Y_f can be finally reduced to

$$Y_f = (\delta p - \delta\bar{p})^T G_{pp}(\delta p - \delta\bar{p}). \tag{8.37}$$

Y_o, from Eq. (8.27), can also be written as

$$\begin{aligned} Y_o &= \delta p^T \delta\mu_o + \delta\mu_o^T \delta p + \delta x_o^T \bar{S}\delta x_o + \delta x_o^T U_o V_o^{-1}\delta\Phi \\ &\quad +\delta\Phi^T V_o^{-1} U_o^T \delta x_o - \delta\Phi^T V_o^{-1}\delta\Phi \end{aligned} \tag{8.38}$$

$$\begin{aligned} &= \delta p^T \delta\mu_o + \delta\mu_o^T \delta p + \delta x_o^T \bar{S}\delta x_o + \delta x_o^T N_o^T \delta\psi \\ &\quad \delta\psi^T N_o \delta x_o + \delta x_o^T M_o^T \delta\mu_o + \delta\mu_o^T M_o \delta x_o - \delta\psi^T a_o \delta\psi \\ &\quad -\delta\mu_0^T b_o \delta\psi - \delta\psi^T b_o^T \delta\mu_o - \delta\mu_0^T c_o \delta\mu_o \end{aligned} \tag{8.39}$$

Since $\delta\mu_0 = 0$ and $\delta\psi = 0$, Y_o becomes

$$Y_o = \delta x_o^T \bar{S}_o \delta x_o \tag{8.40}$$

Therefore, the second variation in Eq. (8.26) becomes

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_o^T \bar{S}_o \delta x_o + (\delta p - \delta\bar{p})^T G_{pp} (\delta p - \delta\bar{p}) \\ &+ \int_0^1 \left[\begin{pmatrix} \delta u^T - \delta\bar{u}^T \end{pmatrix} \begin{pmatrix} \delta p^T - \delta\bar{p}^T \end{pmatrix} \right] \\ &\begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta\bar{u}) \\ (\delta p - \delta\bar{p}) \end{bmatrix} d\tau. \end{aligned} \tag{8.41}$$

For an admissible comparison path, that is one which lies in the neighborhood of the extremal and which satisfies all the constraints, $\delta x_o = 0$, $\delta\psi = 0$, and $\delta\mu_o = 0$. Therefore, if $\bar{S}_o$ and $U_o V_o^{-1}$ are finite over $[0, 1)$, the initial point term outside the integral in Eq. (8.29) vanishes and the second variation reduces to

$$\begin{aligned} \delta^2 \bar{J} &= (\delta p - \delta\bar{p})^T G_{pp} (\delta p - \delta\bar{p}) + \int_0^1 \left[\begin{pmatrix} \delta u^T - \delta\bar{u}^T \end{pmatrix} \begin{pmatrix} \delta p^T - \delta\bar{p}^T \end{pmatrix} \right] \\ &\begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & H_{pp} \end{bmatrix} \begin{bmatrix} (\delta u - \delta\bar{u}) \\ (\delta p - \delta\bar{p}) \end{bmatrix} d\tau. \end{aligned} \tag{8.42}$$

Hence, if Eq. (8.42) is positive definite, the second variation is positive for arbitrary variations δp and δu, unless $\delta p = \delta\bar{p}$ and $\delta u = \delta\bar{u}$. This can occur only if the admissible comparison path is a neighboring extremal. However, Eq. (3.69) indicates that for $\bar{S}_o$ and $U_o V_o^{-1}$ finite and $\delta x_o = 0$, $\delta\mu = 0$, and $\delta\psi = 0$, this results in $\delta\lambda_o = 0$. With this, Eq. (3.77) reduces to $\delta p = 0$ and $\delta u = 0$. Therefore, Eqs. (3.16) - (3.18) simplify to $\delta x = \delta\lambda = \delta\mu = 0$, which implies that $\delta p = 0$, or that there is no admissible comparison path which is a neighboring extremal path. Therefore, the extremal path is a minimum.

On the other hand, if $\bar{S}_o$ and $U_o V_o^{-1}$ become infinite, $\delta x_o = 0$, $\delta\mu = 0$, and $\delta\psi = 0$ could lead to a finite $\delta\lambda_o$ and to an admissible comparison path

which is a neighboring extremal. Therefore, the second variation could become negative and the extremal path is not a minimum. This is also the case when $\bar{S}$ and UV^{-1} become infinite at some point within the interval. If this happens, this point is called a conjugate point.

The sufficient conditions are formulated from the operations needed to form $\delta^2\bar{J}$ as a perfect square. In addition, since a perfect differential is added to the second variation, $\bar{S}$, UV^{-1}, and V^{-1} must be finite for a finite δx and $\delta\Phi$ to lead to a finite $\delta\nu$ and δp, as well as to a finite $\delta\lambda$. These conditions can also be stated as requiring the derivatives of $\bar{S}$, UV^{-1}, and V^{-1} to be integrable over $[0,1)$ or as $\bar{S}$, UV^{-1}, and V^{-1} to be finite over $[0,1)$. The following theorem states the sufficient conditions.

Theorem 2 *If the first variation necessary conditions for a minimum are satisfied and the following conditions are satisfied,*

1) $H_{uu} > 0$,

2) $\begin{bmatrix} H_{uu} & H_{up} \\ H_{pu} & G_{pp} + H_{pp} \end{bmatrix} > 0$,

3) $\bar{S}$ *finite over* $0 \leq \tau < 1$,

4) UV^{-1} *finite over* $0 \leq \tau < 1$, *and*

5) V^{-1} *finite over* $0 \leq \tau < 1$,

then this solution is a minimum.

Chapter 9

Conclusion

Sufficient conditions for optimal control problems with parameterized controls are developed. These conditions have been applied to a fairly realistic missile intercept problem.

In Chapter 2, the first variation necessary conditions for optimal control problems with parameters are developed. The quantity μ, whose differential equation has the same form as the differential equation for λ, is introduced.

In Chapter 3, the theory associated with neighboring extremal trajectories is developed. This is done using the sweep method. Variations of the first variation necessary conditions are taken and this results in a set of Riccati equations which govern the behavior of these trajectories. The neighboring extremal parameterized control law is also developed.

In Chapter 4, the sufficient conditions for a minimum for parameterized control are derived. First, the second variation is obtained by taking the variation of the first variation. The key to the sufficient conditions is the addition of a particular perfect differential to the second variation. This reduces the second variation to a perfect square involving the same form as the neighboring extremal parameters obtained in Chapter 3. The other terms are solely functions of quantities at the initial time. Using these results, the sufficient conditions for a minimum are formulated.

In Chapter 5, the sufficient conditions are applied to a missile intercept problem whose model is an EMRAAT class missile, with a single thrust phase. Optimal trajectories are generated for two realistic missile engagement scenarios, the first one involving a target moving toward the launching aircraft at 1,000 ft/sec at an initial range of 40 miles, and the second involving a target moving away from the launching aircraft at 1,000 ft/sec at an initial range of 30 miles. The characteristics of these optimal trajectories are presented. The sufficient conditions for a minimum are applied to both these cases and are satisfied.

In Chapter 6, the first variation necessary conditions are obtained for the case where there are both parameterized and nonparameterized controls. The first integral is also obtained.

Chapter 7 contains the development of the neighboring extremal trajectories for the class of problems where there are both parameterized and nonparameterized controls. In addition, the neighboring extremal parameterized and nonparameterized control laws are obtained.

In Chapter 8, the second variation condition is obtained for the case where there are both parameterized and non-parameterized controls. The development closely follows that in Chapter 4, and the conditions obtained are analogous to those obtained in Chapter 4.

Further Work

While the sufficient conditions, which are derived, are quite general and encompass a large class of deterministic optimal control problems, they need to be applied to a wider variety of problems to gain experience with the application of these conditions and investigate the behavior of these new Riccati equations. In a similar vein, a 'feedback' control law for parameterized

control needs to be analyzed for neighboring extremal guidance type applications.

Appendix A

Derivation of the Differential Equations for $\bar{S}$, V^{-1}, and UV^{-1} for Parameterized Control

In this Appendix, the differential equations for the components of $\bar{S}$, V^{-1}, and UV^{-1} are derived. First, recall that V and V^{-1} have been defined to be

$$V \triangleq \begin{bmatrix} 0 & n(\tau) \\ n^T(\tau) & \alpha(\tau) \end{bmatrix} \tag{A.1}$$

$$V^{-1} \triangleq \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}. \tag{A.2}$$

The differential equation for V, using Eqs. (3.62)-(3.61), is

$$V' = -\begin{bmatrix} 0 & R^T\tilde{D} \\ \tilde{D}^T R & m^T\tilde{D} + \tilde{D}^T m + \tilde{F} \end{bmatrix}. \tag{A.3}$$

With this equation, the differential equation for V^{-1} is expressed as

$$V^{-1\prime} = V^{-1}\begin{bmatrix} 0 & R^T\tilde{D} \\ \tilde{D}^T R & m^T\tilde{D} + \tilde{D}^T m + \tilde{F} \end{bmatrix} V^{-1}. \tag{A.4}$$

However, the quantity UV^{-1} is expressed as follows

$$UV^{-1} = \begin{bmatrix} Ra + mb^T & Rb + mc \end{bmatrix} = \begin{bmatrix} N^T & M^T \end{bmatrix}. \tag{A.5}$$

Using this equation along with Eq. (A.2), Eq. (A.4) is expressed as

$$\begin{aligned} V^{-1\prime} \;=\; & \left[\begin{matrix} (aR^T + bm^T)\tilde{D}b^T + b\tilde{D}^T(Ra + mb^T) + b\tilde{F}b^T \\ (b^T R^T + cm^T)\tilde{D}b^T + c\tilde{D}^T(Ra + mb^T) + c\tilde{F}b^T \end{matrix} \right. \\ & \left. \begin{matrix} b\tilde{D}^T(Rb + mc) + (aR^T + bm^T)\tilde{D}c + b\tilde{F}c \\ (b^T R^T + cm^T)\tilde{D}c + c\tilde{D}^T(Rb + mc) + c\tilde{F}c \end{matrix} \right] \end{aligned} \tag{A.6}$$

which further reduces to

$$V^{-1\prime} = \begin{bmatrix} N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T & b\tilde{D}^T M^T + N\tilde{D}c + b\tilde{F}c \\ M\tilde{D}b^T + c\tilde{D}^T N^T + c\tilde{F}b^T & M\tilde{D}c + c\tilde{D}^T M^T + c\tilde{F}c \end{bmatrix}. \tag{A.7}$$

The differential equations for a, b, and c are expressed as

$$a' = N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T \tag{A.8}$$

$$b' = b\tilde{D}^T M^T + N\tilde{D}c + b\tilde{F}c \tag{A.9}$$

$$c' = M\tilde{D}c + c\tilde{D}^T M^T + c\tilde{F}c \tag{A.10}$$

or alternatively as

$$a' = (aR^T + bm^T)\tilde{D}b^T + b\tilde{D}^T(Ra + mb^T) + b\tilde{F}b^T \tag{A.11}$$

$$b' = b\tilde{D}^T(Rb + mc) + (aR^T + bm^T)\tilde{D}c + b\tilde{F}c \tag{A.12}$$

$$c' = (b^T R^T + cm^T)\tilde{D}c + c\tilde{D}^T(Rb + mc) + c\tilde{F}c. \tag{A.13}$$

Recall that U was defined to be

$$U = \begin{bmatrix} R & m \end{bmatrix} \tag{A.14}$$

and upon differentiating and using Eqs. (3.58) and (3.59) becomes

$$U' = \begin{bmatrix} -\tilde{A}^T R & -\tilde{A}^T m - (\tilde{E} + S\tilde{D}) \end{bmatrix} \tag{A.15}$$

which reduces to

$$U' = -\tilde{A}^T U - \begin{bmatrix} 0 & (\tilde{E} + S\tilde{D}) \end{bmatrix}. \tag{A.16}$$

The differential equation for $\bar{S}$ is obtained using the quantities derived above. Recall that $\bar{S}$ is defined as

$$\bar{S} = S - UV^{-1}U^T. \tag{A.17}$$

Differentiating this yields

$$\bar{S}' = S' - U'V^{-1}U^T - UV^{-1\prime}U^T - UV^{-1}U^{T\prime}. \tag{A.18}$$

After certain substitutions from Eqs. (3.57), (A.16), and (A.7) are made, this becomes

$$\begin{aligned}
\bar{S}' &= -\tilde{A}^T S - S\tilde{A} - \tilde{C} \\
&\quad + \tilde{A}^T UV^{-1}U^T + \begin{bmatrix} 0 & (\tilde{E} + S\tilde{D}) \end{bmatrix} V^{-1}U^T \qquad \text{(A.19)} \\
&\quad - U \begin{bmatrix} N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T \\ M\tilde{D}b^T + c\tilde{D}^T N^T + c\tilde{F}b^T \end{bmatrix} \\
&\quad \begin{bmatrix} b\tilde{D}^T M^T + N\tilde{D}c + b\tilde{F}c \\ M\tilde{D}c + c\tilde{D}^T M^T + c\tilde{F}c \end{bmatrix} U^T \\
&\quad + UV^{-1}U^T\tilde{A} + UV^{-1} \begin{bmatrix} 0 \\ (\tilde{E}^T + \tilde{D}^T S) \end{bmatrix}. \qquad \text{(A.20)}
\end{aligned}$$

The quantity $UV^{-1}U^T$ is also expressed as

$$UV^{-1}U^T = \begin{bmatrix} N^T & M^T \end{bmatrix} \begin{bmatrix} R^T \\ m^T \end{bmatrix} = N^T R^T + M^T m^T = RN + mM. \tag{A.21}$$

If Eqs. (A.21) are used and terms involving $\bar{S}$ are factored, the equation further reduces to

$$\begin{aligned}
\bar{S}' &= -\tilde{A}^T(S - UV^{-1}U^T) - (S - UV^{-1}U^T)\tilde{A} - \tilde{C} \\
&\quad + \tilde{E}(b^T R^T + cm^T) + (Rb + mc)\tilde{E}^T - (Rb + mc)\tilde{F}(b^T R^T + cm^T) \\
&\quad + (S - UV^{-1}U^T)\tilde{D}M - M^T\tilde{D}^T(S - UV^{-1}U^T). \qquad \text{(A.22)}
\end{aligned}$$

This equation further simplifies to

$$\begin{aligned}
\bar{S}' &= -\tilde{A}^T\bar{S} - \bar{S}\tilde{A} + \tilde{E}M + M^T\tilde{E}^T \\
&\quad + \bar{S}\tilde{D}M + M^T\tilde{D}^T\bar{S} - \tilde{C} - M^T\tilde{F}M. \qquad \text{(A.23)}
\end{aligned}$$

Using similar reasoning, the differential equations for M and N is derived as follows

$$\frac{dUV^{-1}}{d\tau} = U'V^{-1} + UV^{-1\prime} \tag{A.24}$$

$$\begin{aligned} &= -\tilde{A}^T UV^{-1} - \begin{bmatrix} 0 & (\tilde{E} + S\tilde{D}) \end{bmatrix} V^{-1} \\ &\quad + U \begin{bmatrix} N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T & b\tilde{D}^T M^T + N\tilde{D}c + b\tilde{F}c \\ M\tilde{D}b^T + c\tilde{D}^T N^T + c\tilde{F}b^T & M\tilde{D}c + c\tilde{D}^T M^T + c\tilde{F}c \end{bmatrix}. \end{aligned} \tag{A.25}$$

Using the definition of UV^{-1} from Eq. (A.5) and the definition of V^{-1} from Eq. (A.2) and factoring, the differential equations for M and N is expressed as

$$N^{T\prime} = -\tilde{A}^T N^T - \bar{S}\tilde{D}b^T - \tilde{E}b^T + M^T\tilde{D}^T N^T + M^T\tilde{F}b^T \tag{A.26}$$

$$N' = -N\tilde{A} - b\tilde{D}^T\bar{S} - b\tilde{E}^T + N\tilde{D}M + b\tilde{F}M \tag{A.27}$$

$$M^{T\prime} = -\tilde{A}^T M^T - \bar{S}\tilde{D}c + M^T\tilde{D}^T M^T + M^T\tilde{F}c - \tilde{E}c \tag{A.28}$$

$$M' = -M\tilde{A} - c\tilde{D}^T\bar{S} - c\tilde{E}^T + M\tilde{D}M + c\tilde{F}M. \tag{A.29}$$

$$\tag{A.30}$$

In summary, the differential equations for $\bar{S}$, M, N, a, b, and c are as follows

$$\begin{aligned} \bar{S}' &= -\tilde{A}^T\bar{S} - \bar{S}\tilde{A} + \tilde{E}M + M^T\tilde{E}^T \\ &\quad + \bar{S}\tilde{D}M + M^T\tilde{D}^T\bar{S} - \tilde{C} - M^T\tilde{F}M \end{aligned} \tag{A.31}$$

$$M' = -M\tilde{A} - c\tilde{D}^T\bar{S} - c\tilde{E}^T + M\tilde{D}M + c\tilde{F}M \tag{A.32}$$

$$N' = -N\tilde{A} - b\tilde{D}^T\bar{S} - b\tilde{E}^T + N\tilde{D}M + b\tilde{F}M \tag{A.33}$$

$$a' = N\tilde{D}b^T + b\tilde{D}^T N^T + b\tilde{F}b^T \tag{A.34}$$

$$b' = N\tilde{D}c + b\tilde{D}^T M^T + b\tilde{F}c \tag{A.35}$$

$$c' = M\tilde{D}c + c\tilde{D}^T M^T + c\tilde{F}c. \tag{A.36}$$

Appendix B

Derivation of the Differential Equations for $\bar{S}$, V^{-1}, and UV^{-1} for Parameterized and Nonparameterized Control

In this Appendix, the differential equations for the components of $\bar{S}$, V^{-1}, and UV^{-1} are derived. First, recall that V and V^{-1} have been defined to be

$$V \triangleq \begin{bmatrix} Q(\tau) & n(\tau) \\ n^T(\tau) & \alpha(\tau) \end{bmatrix} \tag{B.1}$$

$$V^{-1} \triangleq \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}. \tag{B.2}$$

The differential equation for V, using Eqs.(3.62)-(3.61), is

$$V' = \begin{bmatrix} R^T BR & R^T Bm - R^T D \\ m^T BR - D^T R & m^T Bm - m^T D - D^T m - F \end{bmatrix} \tag{B.3}$$

which is expressed more in a more compact form, using Eq. (3.65), as

$$V' = U^T BU - \begin{bmatrix} 0 & R^T D \\ D^T R & m^T D + D^T m + F \end{bmatrix}. \tag{B.4}$$

With this equation, the differential equation for V^{-1} is expressed as

$$V^{-1\prime} = -V^{-1} U^T BUV^{-1} + V^{-1} \begin{bmatrix} 0 & R^T D \\ D^T R & m^T D + D^T m + F \end{bmatrix} V^{-1}. \tag{B.5}$$

However, the quantity UV^{-1} is expressed as follows

$$UV^{-1} = \begin{bmatrix} Ra + mb^T & Rb + mc \end{bmatrix} = \begin{bmatrix} N^T & M^T \end{bmatrix}. \tag{B.6}$$

If this equation, along with Eq.(B.2), is used, Eq. (B.5) is expressed as

$$\begin{aligned}
V^{-1\prime} &= -\begin{bmatrix} NBN^T & NBM^T \\ MBN^T & MBM^T \end{bmatrix} \\
&+ \left[\begin{matrix} (aR^T + bm^T)Db^T + bD^T(Ra + mb^T) + bFb^T \\ (b^TR^T + cm^T)Db^T + cD^T(Ra + mb^T) + cFb^T \end{matrix} \right. \\
&\left. \begin{matrix} bD^T(Rb + mc) + (aR^T + bm^T)Dc + bFc \\ (b^TR^T + cm^T)Dc + cD^T(Rb + mc) + cFc \end{matrix} \right] \qquad \text{(B.7)}
\end{aligned}$$

which further reduces to

$$\begin{aligned}
V^{-1\prime} &= \left[\begin{matrix} -NBN^T + NDb^T + bD^TN^T + bFb^T \\ -MBN^T + MDb^T + cD^TN^T + cFb^T \end{matrix} \right. \\
&\left. \begin{matrix} -NBM^T + bD^TM^T + NDc + bFc \\ -MBM^T + MDc + cD^TM^T + cFc \end{matrix} \right]. \qquad \text{(B.8)}
\end{aligned}$$

The differential equations for a, b, and c are expressed as

$$\begin{aligned}
a' &= -NBN^T + NDb^T + bD^TN^T + bFb^T \qquad \text{(B.9)} \\
b' &= -NBM^T + NDc + bD^TM^T + bFc \qquad \text{(B.10)} \\
c' &= -MBM^T + MDc + cD^TM^T + cFc \qquad \text{(B.11)}
\end{aligned}$$

or alternatively as

$$\begin{aligned}
a' &= -(aR^T + bm^T)B(Ra + mb^T) + (aR^T + bm^T)Db^T \\
&\quad + bD^T(Ra + mb^T) + bFb^T \qquad \text{(B.12)} \\
b' &= -(aR^T + bm^T)B(Rb + mc) + bD^T(Rb + mc) \\
&\quad + (aR^T + bm^T)Dc + bFc \qquad \text{(B.13)} \\
c' &= -(b^TR^T + cm^T)B(Rb + mc) + (b^TR^T + cm^T)Dc \qquad \text{(B.14)} \\
&\quad + cD^T(Rb + mc) + cFc.
\end{aligned}$$

Recall that U was defined as

$$U = \begin{bmatrix} R & m \end{bmatrix} \qquad \text{(B.15)}$$

and upon differentiating and using Eqs. (3.58) and (3.59) becomes

$$U' = \begin{bmatrix} (SB - A^T)R & (SB - A^T)m - (E + SD) \end{bmatrix} \tag{B.16}$$

which reduces to

$$U' = (SB - A^T)U - \begin{bmatrix} 0 & (E + SD) \end{bmatrix}. \tag{B.17}$$

The differential equation for $\bar{S}$ is obtained using the quantities derived above. Recall that $\bar{S}$ is defined as

$$\bar{S} = S - UV^{-1}U^T. \tag{B.18}$$

Differentiating Eq. (B.18) yields

$$\bar{S}' = S' - U'V^{-1}U^T - UV^{-1\prime}U^T - UV^{-1}U^{T\prime}. \tag{B.19}$$

If Eqs.(3.57), (B.17), and (B.8) are used, this becomes

$$\begin{aligned} \bar{S}' &= SBS - A^TS - SA - C \\ &\quad -(SB - A^T)UV^{-1}U^T + \begin{bmatrix} 0 & (E + SD) \end{bmatrix} V^{-1}U^T \qquad \text{(B.20)} \\ &\quad -U \begin{bmatrix} -NBN^T + NDb^T + bD^TN^T + bFb^T \\ -MBN^T + MDb^T + cD^TN^T + cFb^T \end{bmatrix. \\ &\qquad \left. \begin{matrix} -NBM^T + bD^TM^T + NDc + bFc \\ -MBM^T + MDc + cD^TM^T + cFc \end{matrix} \right] U^T \\ &\quad -UV^{-1}U^T(BS - A) + UV^{-1} \begin{bmatrix} 0 \\ (E^T + D^TS) \end{bmatrix}. \qquad \text{(B.21)} \end{aligned}$$

The quantity $UV^{-1}U^T$ is expressed as

$$UV^{-1}U^T = \begin{bmatrix} N^T & M \end{bmatrix} \begin{bmatrix} R^T \\ m^T \end{bmatrix} = N^TR^T + M^Tm^T = RN + mM. \tag{B.22}$$

If Eqs. (B.22) are used and terms involving $\bar{S}$ are factored, the equation is further reduced to

$$\bar{S}' = (S - UV^{-1}U^T)B(S - UV^{-1}U^T) - A^T(S - UV^{-1}U^T)$$

$$-(S-UV^{-1}U^T)A - C + E(b^TR^T + cm^T) + (Rb+mc)E^T$$
$$-(Rb+mc)F(b^TR^T+cm^T) + (S-UV^{-1}U^T)DM$$
$$-M^TD^T(S-UV^{-1}U^T). \qquad \text{(B.23)}$$

This equation further simplifies to

$$\begin{aligned}\bar{S}' &= \bar{S}B\bar{S} - A^T\bar{S} - \bar{S}A + EM + M^TE^T \\ &\quad + \bar{S}DM + M^TD^T\bar{S} - M^TFM - C. \end{aligned} \qquad \text{(B.24)}$$

Using similar reasoning, the differential equations for M and N is derived as follows

$$\frac{dUV^{-1}}{d\tau} = U'V^{-1} + UV^{-1\prime} \qquad \text{(B.25)}$$

$$\begin{aligned} &= (SB - A^T)UV^{-1} - \begin{bmatrix} 0 & (E+SD) \end{bmatrix} V^{-1} \\ &\quad + U \begin{bmatrix} -NBN^T + NDb^T + bD^TN^T + bFb^T & -NBM^T + bD^TM^T + NDc + bFc \\ -MBN^T + MDb^T + cD^TN^T + cFb^T & -MBM^T + MDc + cD^TM^T + cFc \end{bmatrix}. \end{aligned} \qquad \text{(B.26)}$$

If the definition of UV^{-1} from Eq. (B.6) and the definition of V^{-1} from Eq. (B.2) is used and Eq. (B.26) is factored, the differential equations for M and N are expressed as

$$N^{T\prime} = (\bar{S}B - A^T)N^T - \bar{S}Db^T - Eb^T + M^TD^TN^T + M^TFb^T \qquad \text{(B.27)}$$
$$N' = N(B\bar{S} - A) - bD^T\bar{S} - bE^T + NDM + bFM \qquad \text{(B.28)}$$
$$M^{T\prime} = (\bar{S}B - A^T)M^T - \bar{S}Dc + M^TD^TM^T + M^TFc - Ec \qquad \text{(B.29)}$$
$$M' = M(B\bar{S} - A) - cD^T\bar{S} - cE^T + MDM + cFM. \qquad \text{(B.30)}$$

In summary, the differential equations for $\bar{S}$, M, N, a, b, and c are as follows

$$\bar{S}' = \bar{S}B\bar{S} - A^T\bar{S} - \bar{S}A + EM + M^TE^T$$

$$
\begin{aligned}
& +\bar{S}DM + M^T D^T \bar{S} - M^T FM - C && \text{(B.31)} \\
M' &= M(B\bar{S} - A) - cD^T\bar{S} - cE^T + MDM + cFM && \text{(B.32)} \\
N' &= N(B\bar{S} - A) - bD^T\bar{S} - bE^T + NDM + bFM && \text{(B.33)} \\
a' &= -NBN^T + NDb^T + bD^T N^T + bFb^T && \text{(B.34)} \\
b' &= -NBM^T + NDc + bD^T M^T + bFc && \text{(B.35)} \\
c' &= -MBM^T + MDc + cD^T M^T + cFc. && \text{(B.36)}
\end{aligned}
$$

Appendix C

The Parameterized Shooting Algorithm

C.1 Introduction

The parameterized shooting method is a technique which solves the linear two point boundary value problem when the control is parameterized. In general, the final time, when it is unknown, is included as part of the parameters. This particular algorithm uses the transition matrix approach to solve for the parameters. As in the regular shooting method, it requires the guesses of the Lagrange multipliers at the initial time as well as the guesses of the parameters.

C.2 The Pertinent Equations

The equations of interest are the first variation necessary conditions obtained in Section 2.2. They are as follows

$$x' = f(\tau, x, u_i, t_f) \tag{C.1}$$

$$\lambda' = -H_x^T(\tau, x, u_i, t_f, \lambda) \tag{C.2}$$

$$\mu_{u_i}{}' = -H_{u_i}^T(\tau, x, u_i, t_f, \lambda) \tag{C.3}$$

$$\mu_{t_f}{}' = -H_{t_f}(\tau, x, u_i, t_f, \lambda) \tag{C.4}$$

with the boundary conditions

$$\psi(x_f, t_f) = 0 \tag{C.5}$$

$$\lambda_{x_f} = G_{x_f}^T \tag{C.6}$$

$$\mu_{u_{i_f}} = 0 \tag{C.7}$$

$$\mu_{t_{f_f}} = G_{t_f}. \tag{C.8}$$

If the vector z is defined as follows

$$z = \begin{bmatrix} x \\ \lambda \\ \mu_{u_i} \\ \mu_{t_f} \end{bmatrix}, \tag{C.9}$$

Eqs.(C.1)-(C.4) become

$$z' = F(\tau, z, u_i, t_f), \tag{C.10}$$

where

$$F(\tau, z, u_i, t_f) = \begin{bmatrix} f(\tau, z, u_i, t_f) \\ -H_x^T(\tau, z, u_i, t_f) \\ -\left(\frac{\partial u}{\partial u_i}\right)^T H_u^T(\tau, z, u_i, t_f) \\ -H_{t_f}(\tau, z, u_i, t_f) \end{bmatrix}. \tag{C.11}$$

Taking the variation of Eq. (C.10) yields

$$\delta z' = F_z \delta z + F_{u_i} \delta u_i + F_{t_f} \delta t_f. \tag{C.12}$$

A solution of Eq.(C.12) of the following form is assumed

$$\delta z = \Phi \delta z_0 + \zeta \delta u_i + K \delta t_f, \tag{C.13}$$

which upon differentiation becomes

$$\delta z' = \Phi' \delta z_0 + \zeta' \delta u_i + K' \delta t_f. \tag{C.14}$$

If Eq. (C.13) is substituted for δz into Eq. (C.12), it results in

$$\delta z' = F_z \Phi \delta z_0 + (F_z \zeta + F_{u_i}) \delta u_i + \left(F_z K + F_{t_f}\right) \delta t_f. \tag{C.15}$$

When Eqs (C.14) and (C.15) are equated, the following differential equations for Φ, ζ, and K are obtained

$$\Phi' = F_z\Phi; \qquad \Phi_0 = I \tag{C.16}$$

$$\zeta' = F_z\zeta + F_{u_i}; \qquad \zeta_0 = 0 \tag{C.17}$$

$$K' = F_zK + F_{t_f}; \qquad K_0 = 0. \tag{C.18}$$

As stated in Eq. (C.16), the state transition matrix behaves according to the following relation

$$\Phi(\tau,\tau_0)' = F_z\Phi(\tau,\tau_0) \tag{C.19}$$

where F_z is expressed as

$$F_z = \begin{bmatrix} f_x & 0 & 0 & 0 \\ -H_{xx} & -H_{x\lambda} & 0 & 0 \\ -\left(\frac{\partial u}{\partial u_i}\right)^T H_{ux} & -\left(\frac{\partial u}{\partial u_i}\right)^T H_{u\lambda} & 0 & 0 \\ -H_{t_fx} & -H_{t_f\lambda} & 0 & 0 \end{bmatrix} \tag{C.20}$$

The state transition matrix reduces to the following

$$\Phi(\tau,\tau_0) = \begin{bmatrix} \Phi_{11} & 0 & 0 & 0 \\ \Phi_{21} & \Phi_{22} & 0 & 0 \\ \Phi_{31} & \Phi_{32} & I & 0 \\ \Phi_{41} & \Phi_{42} & 0 & I \end{bmatrix} \tag{C.21}$$

where the individual sub-matrices of Φ can be obtained from

$$\Phi'_{11} = f_x\Phi_{11}; \qquad \Phi_{11_0} = I \tag{C.22}$$

$$\Phi'_{21} = -H_{xx}\Phi_{11} + -H_{x\lambda}\Phi_{21}; \qquad \Phi_{21_0} = 0 \tag{C.23}$$

$$\Phi'_{22} = -H_{x\lambda}\Phi_{22}; \qquad \Phi_{22_0} = I \tag{C.24}$$

$$\Phi'_{31} = -\left(\frac{\partial u}{\partial u_i}\right)^T H_{ux}\Phi_{11} - \left(\frac{\partial u}{\partial u_i}\right)^T H_{u\lambda}\Phi_{21}; \qquad \Phi_{31_0} = 0 \tag{C.25}$$

$$\Phi'_{32} = -\left(\frac{\partial u}{\partial u_i}\right)^T H_{u\lambda}\Phi_{22}; \qquad \Phi_{32_0} = 0 \tag{C.26}$$

$$\Phi'_{41} = -H_{t_fx}\Phi_{11} + -H_{t_f\lambda}\Phi_{21}; \qquad \Phi_{41_0} = 0 \tag{C.27}$$

$$\Phi'_{42} = -H_{t_f\lambda}\Phi_{22}; \qquad \Phi_{42_0} = 0. \tag{C.28}$$

It turns out that the only components of the state transition matrix which are needed are Φ_{22}, Φ_{32}, and Φ_{42}.

For a problem with i states, j controls, k parameters, and l constraints, h, which contains the conditions at the final time which need to satisfied, is of dimension $i+k$. At the initial time, there are $i+k$ quantities which are known; only λ_0 is unknown. At the terminal time, there are $l+i+k$ final conditions. Of these, l equations are solved for the l Lagrange multipliers ν, which are then eliminated from the remaining conditions to yield the vector h which can be written as

$$h(z_f, u_i, t_f) = 0. \tag{C.29}$$

The two point boundary value problem is considered solved when $h(z_f, u_i, t_f) = 0$. This usually involves an iterating process until this condition is satisfied. During the solution process δh can be approximated by

$$\begin{aligned} \delta h &\approx h_k - h_{k-1} = \eta h_{k-1} - h_{k-1} = -(1-\eta) h_{k-1} && \text{(C.30)} \\ &\approx -\alpha h_{k-1} && \text{(C.31)} \end{aligned}$$

where h_k is the value of h at iteration k, h_{k-1} is the corresponding value at iteration $k-1$, and α and η are positive quantities which are less than 1. The variation of h is

$$\delta h = h_{z_f} \delta z_f + h_{u_i} \delta u_i + h_{t_f} \delta t_f \tag{C.32}$$

which using Eq. (C.13) evaluated at the final time becomes

$$\delta h = h_{z_f} \Phi_f \delta z_0 + \left(h_{z_f} \zeta_f + h_{u_i} \right) \delta u_i + \left(h_{z_f} K_f + h_{t_f} \right) \delta t_f. \tag{C.33}$$

At the initial time, since x is known, $\mu_{u_i} = 0$, and $\mu_{t_f} = 0$, δz_0 is merely

$$\delta z_0 = \begin{bmatrix} 0 \\ \delta \lambda_0 \\ 0 \\ 0 \end{bmatrix}, \tag{C.34}$$

hence

$$\Phi_f \delta z_0 = \Phi_{2_f} \delta\lambda_0. \tag{C.35}$$

In most problems, the control does not appear in the end-point function, G; therefore,

$$h_{u_i} = 0. \tag{C.36}$$

Using Eqs.(C.35) and (C.36), Eq.(C.33) can be rewritten in matrix form as

$$-\alpha h_{k-1} = \begin{bmatrix} h_{z_f}\Phi_{2_f} & h_{z_f}\zeta_f & h_{z_f}K_f + h_{t_f} \end{bmatrix} \begin{bmatrix} \delta\lambda_0 \\ \delta u_i \\ \delta t_f \end{bmatrix} \tag{C.37}$$

where α is chosen to ensure that $||h||$ decreases after each iteration. As stated earlier, h is of dimension $i+k$; the number of unknowns is $i+k$ and are obtained by solving Eq. (C.37).

C.3 The Algorithm

The parameterized shooting algorithm is as follows:

1. Guess λ_0, u_i, and t_f

2. Integrate from $\tau = 0$ to $\tau = 1$

$$z' = \begin{bmatrix} f(\tau, z, u_i, t_f) \\ -H_x^T(\tau, z, u_i, t_f) \\ -\left(\frac{\partial u}{\partial u_i}\right)^T H_u^T(\tau, z, u_i, t_f) \\ -H_{t_f}(\tau, z, u_i, t_f) \end{bmatrix}; \qquad z_0 = \begin{bmatrix} x_0 \\ \lambda_0 \\ 0 \\ 0 \end{bmatrix} \tag{C.38}$$

$$\Phi'_{22} = -H_{x\lambda}\Phi_{22}; \qquad \Phi_{22_0} = I \tag{C.39}$$

$$\Phi'_{32} = -\left(\frac{\partial u}{\partial u_i}\right)^T H_{u\lambda}\Phi_{22}; \qquad \Phi_{32_0} = 0 \tag{C.40}$$

$$\Phi'_{42} = -H_{t_f\lambda}\Phi_{22}; \qquad \Phi_{42_0} = 0 \tag{C.41}$$

$$\zeta' = F_z\zeta + F_{u_i}; \qquad \zeta_0 = 0 \tag{C.42}$$

$$K' = F_z K + F_{t_f}; \qquad K_0 = 0. \tag{C.43}$$

3. Evaluate h and $||h||$. If $||h|| \leq \epsilon$, stop; if not, go to Step 4.

$$h(z_f, u_i, t_f) = \begin{bmatrix} \psi(x_f, t_f) \\ \lambda_f - G_{x_f}^T \\ \mu_{u_{i_f}} \\ \mu_{t_{f_f}} - G_{t_f} \end{bmatrix}. \tag{C.44}$$

4. Calculate h_{z_f} and h_{t_f}.

5. Compute the correction terms to the guesses by solving the linear system

$$\begin{bmatrix} h_{z_f}\Phi_{2_f} & h_{z_f}\zeta_f & h_{z_f}K_f + h_{t_f} \end{bmatrix} \begin{bmatrix} \delta\lambda_0 \\ \delta u_i \\ \delta t_f \end{bmatrix} = -\alpha h_k. \tag{C.45}$$

Choose α by either percent correction or scaling to ensure that $||h||_{k+1} < ||h||_k$.

6. Update λ_0, u_i, t_f; go to Step 2.

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